Flip Dynamics, Structure of Tiling Spaces.

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Flips

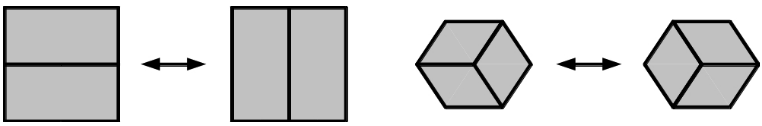


FIGURE – a domino flip and an lozenge fllip

Flip : local transformation of a tiling involving a few tiles.

In this lecture, we will work with domino tilings,
but lozenge tilings can be treated in a similar way.

Domain

Domain : finite simply connected (i.e. with no hole) union of cells of the square lattice.
The boundary of D is a unique cycle

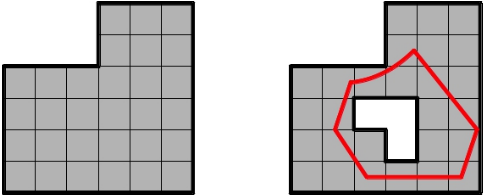


FIGURE – Left : a domain Right : a non simply connected region

Our goal is to study the effect of flips on the set of tilings of a fixed domain D.

Tiling Space

Tiling space of D : the undirected graph

- whose vertex set is the set of tilings of D,
- the pair (T, T') is an edge if one can pass from T to T' by a single flip.

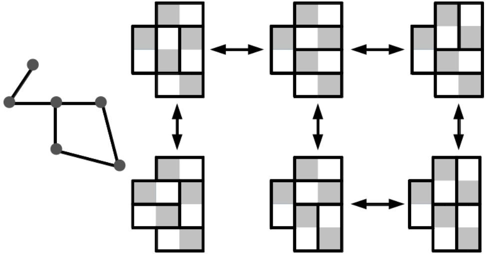


FIGURE – A tiling space

Question : What about the structure of tiling spaces ?

A tool : path value

Direct edges of the square lattice, according to cell colorings (white in the left side, black on the right side).

Definition

- $\delta_h(v, v') = 1$ if (v, v') is directed as said above,
- $\delta_h(v, v') = -1$ otherwise (i.e. if $\delta_h(v', v) = 1$).

By extension, for each path P of $\mathbb{Z}^2$,

$$\delta_h(P) = \sum_{(v,v') \text{ is an edge of } P} \delta_h(v, v')$$

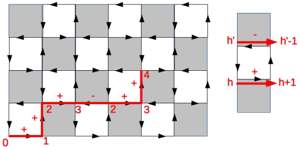


Figure – Computation of $\delta_h(P)$.

Cycle value

Lemma : Let C be a counterclockwise elementary cycle. Let W_C denote the number of white cells inside P and B_C denote the number of white cells inside P. We have

$$\delta_h(C) = 4(W_C - B_C).$$

Proof : by induction, or by the "camel arm" property

....

Corollary : If the cycle C follows the boundary of a tiled domain, then $\delta_h(C) = 0$.
In particular the value $\delta_h(C)$ of a cycle C around a single domino is null.

Height Function of a Tiling : Definition

Corollary : If P and P' are paths with the same endpoints, and cut no tile, then $\delta_h(P) = \delta_h(P')$.

Definition : For each tiling T of a domain D, and each vertex v,

$$h_T(v) = \delta(P_{(O \to v, T)})$$

where $P_{(O \to v, T)}$ denotes any path, from a fixed vertex O of the boundary of D to v, which cuts no tile in T.

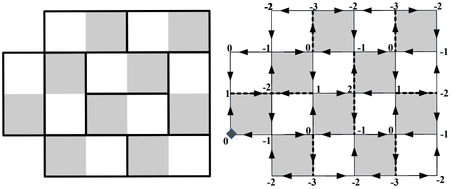


FIGURE – From a tiling to its height function

Height Function of a Tiling : Directing Flips.

Remark : If v is on the boundary of D, then the value $h_T(v)$ does not depend on the tiling T.

Remark : If T and T' only differ by a single flip located in v, then

- $h_T(v') = h_{T'}(v')$ for $v' \neq v$,
- $|h_T(v) - h_{T'}(v)| = 4$.

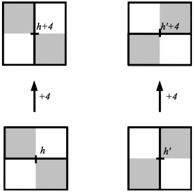


FIGURE – upwards flips

This allows to give an orientation to flips.

Height Function of a Tiling : Directed Tiling Space

The tiling space becomes a **directed acyclic graph**.

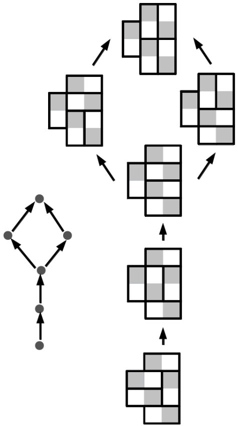


FIGURE – Tiling space with edges directed by height functions

Height Function of a Tiling : Local Characterization

Proposition : (local characterization) Let h be a function $V \to \mathbb{Z}$. there exists a tiling T such that $h = h_T$ if and only if :

- $f(O) = 0$,
- for each (well) directed edge (v, v')
  either $h(v') = h(v) + 1$ or $h(v') = h(v) - 3$,
- for each (well) directed edge (v, v') such that $[v, v']$ is on the boundary of D, $h(v') = h(v) + 1$.

$\Longrightarrow$ is obvious,

$\Longleftarrow$ ....

Corollary : The value $h_T(v) \mod [4]$ does not depends on the tiling T of D.
Moreover, for v on the boundary of D the value $h_T(v)$ does not depends on the tiling T of D.

Height Function of a Tiling : Flip Interpretation

Proposition : let T and T' be two tilings of D. The following conditions are equivalent :

1. $h_T \leq h_{T'}$ (i.e. for each vertex v of D, $h_T(v) \leq h_{T'}(v)$).
2. There exists a finite sequence ($T = T_0, T_1, ..., T_p = T'$) such that, for each $i < p$, one can pass from T_i to T_{i+1} by a single upward flip.

Proof :

(2) $\implies$ (1) is obvious,

(1) $\implies$ (2)

....

Lattice Structure

Applying the proposition of local characterization, one gets the following :

Proposition : (Lattice structure) let T and T' be two tilings of D. There exists

- a tiling $T_{\min}$ such that $h_{T_{\min}} = \min(h_T, h_{T'})$,
- a tiling $T_{\max}$ such that $h_{T_{\max}} = \max(h_T, h_{T'})$.

...

Summary before Applications

- Tilings $\Longleftrightarrow$ Locally Characterized Height functions
- Partial order : tilings are canonically ordered by height functions
- The order can be interpreted with flips.
- The order confers to the tiling space a structure of distributive lattice.

Now, we can turn towards applications.

Flip Connectivity

- From the lattice structure, the space tiling admits a global minimal tiling T_0.
- From the geometrical interpretation, for any tiling T, there exists a sequence of upward flips to pass from T_0 to T.

Thus :

Proposition : The tiling space is connected : for any pair (T, T') of tilings, one can pass from T to T' be a sequence of flips.

Tiling Algorithm (Preliminaries)

Question : given a domain D, how to compute a tiling of D (or claim that there is no tiling) ?

Idea : compute the minimal tiling T_0.

Lemma (Convexity Lemma) : If v is not on the boundary of D, then there exists an edge (v, v') of D such that

$$h_{T_0}(v') = h_{T_0}(v) + 1.$$

Proof : Since, otherwise, a downward flip can be done. Contradiction

...

Corollary : Let $M_0 = \max\{h_{T_0}(v),\ v \in D\}$. If $M_0 = h_{T_0}(v)$, then v is the boundary of D.

Tiling Algorithm (Realization)

The minimal tiling T_0 can be constructed from the top to the bottom, "slice by slice". For $M \in \mathbb{Z}$, let $V_M = \{v', h_{T_0}(v') \geq M\}$.

Initialization : construct h_{T_0} on the boundary of D and V_{M_0}.

Loop : Assume that h_{T_0} is constructed on V_M.

Put a domino in front of each vertex v such that $h_{T_0}(v) = M$, (and h_{T_0} is for each neighbor of v).

This allows to construct h_{T_0} on V_{M-1}.

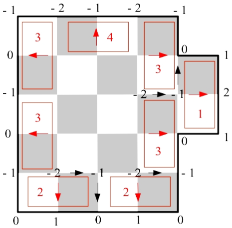


FIGURE – Successive constructions of h_{T_0} on V_2, V_0, V_{-1} and V_{-2}

Tiling Algorithm (a Failure case)

There is no tiling when a contradiction appears for somme value.

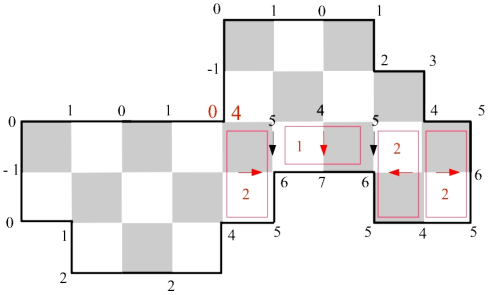


FIGURE – A case when the algorithm detects an impossibility.
It appears an edge (v, v') such that the height difference is at least 4 (in absolute value)

Distance between two Tilings

Proposition : let $d(T, T')$ be the minimal number of flips to pass from T to T'. We have :

$$d(T, T') = \frac{1}{4} \sum_{v} |h_T(v) - h_{T'}(v)|$$

The inequality : $d(T, T') \geq \frac{1}{4} \sum_v |h_T(v) - h_{T'}(v)|$ is obvious.

For the inequality $d(T, T') \leq \frac{1}{4} \sum_v |h_T(v) - h_{T'}(v)|$:

$$d(T, T') \leq d(T, T_{\min}) + d(T_{\min}, T')$$

...

Group Presentation

A group G can possibly be defined by :

- a finite set $S = \{a, b, ...\}$ of *generators* (letters),
- a finite set $R = \{r_1, r_2, ..., r_n\}$ of *relators* (finite words on the alphabet $\{a, b, ..., a^{-1}, b^{-1}, ...\}$)

The group $G =< S \,|\, R >$ is the unique one such that

- each element $g \in G$ can be expressed as a sequence of elements of $\{a, b, ..., a^{-1}, b^{-1}, ...\}$
- all relators express the identity 1_G of G,
- each true equality in the group can be deduced from equalities $r_1 = r_2 = ... = r_n = 1_G$.

Examples :

- $\mathbb{Z}/p\mathbb{Z} =< a \,|\, a^p >$,
- $\mathbb{Z}^2 =< a, b \,|\, aba^{-1}b^{-1} >$

Cayley Graphs

Given a presentation $< S|R >$ of a group G, the *Cayley graph* $G_{<S|R>}$ associated is the directed graph whose vertices are the element of G, and there is an arc form g to g' labeled by the generator a if $ga = g'$.

Examples :

- $< a \,|\, a^5 >$: Directed Cycle C_5,
- $< a, b \,|\, aba^{-1}b^{-1} >$: Square Grid,
- $< a, b, c \,|\, abc, acb >$: Triangular Grid,
- $< a, b \,|\, a^5, b^2, (ab)^3 >$: *Try to guess what it looks like*

All relators correspond to cycles in the graph, and each cycle in the graph is a combination of cycles given by relators.

Some Remarks about Group Presentations

Remark : A group can have several presentations. The same group can lead to different Cayley Graphs
Example : triangular and square grids both are Cayley Graphs of $\mathbb{Z}^2$.

Remark : Let $G =< S \mid r_1, r_2, ..., r_p >$, $G' =< S \mid r'_1, r'_2, ...r'_{p'} >$ be two group presentations, and, assume that, according to rules of group computing, we have :

$$r'_1 = r'_2 = ...r'_{p'} = 1 \implies r_1 = r_2 = ...r_p = 1$$

Then there exists a canonical surjective morphism $\phi : G \to G'$.

Undecidability and Group Presentations

Word Problem
Input : a generator set S, a relator set R and a word u on the alphabet $S \cup S^{-1}$.
Question : is the equality $u = 1$ true in $< S \mid R >$?

Trivial Group Problem
Input : a generator set S, a relator set R
Question : is $< S \mid R >$ the trivial group ?

Result : These two problems are undecidable

Group Function of a tiling

Given a set of tiles, a tiling group is the group G given by the presentation $< S|R >$, where

- S is a set of elementary moves in the grid in which tiles occur,
- R is the set of contour words of tiles

Examples :
Domino Group : $< a, b|ab^2a^{-1}b^{-2}, a^2ba^{-2}b^{-1} >$
Lozenge Group : $< a, b, c|aba^{-1}b^{-1}, aca^{-1}c^{-1}, bcb^{-1}c^{-1} >$ $(= \mathbb{Z}^3)$.

Proposition : There is a canonical surjective mapping :

$$\textit{Tiling Group} \rightarrow \textit{Grid}.$$

Group Function

Proposition : Given a tiling T of a domain D, and an origin vertex O on its boundary, there exists a unique mapping

$$f_T : D \to G,$$

such that

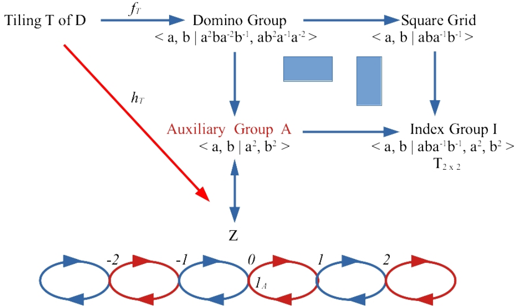
- $f_T(O) = 1_G$,
- for each edge $(v, v') \in D$, and each move x, such that
  - $v\,x = v'$,
  - $[v, v']$ cuts no tile of T,

  we have

$$f_T(v)\,x = f_T(v').$$

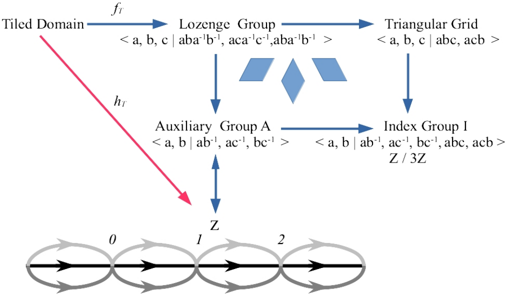
From Group Function to Height Function for Dominoes

Idea : The tiling group is not tractable, but a simpler group contains a sufficient information to encode the tiling.



Remark : the index I ensures that $h_T(v) = h_{T'}(v) \mod 4$

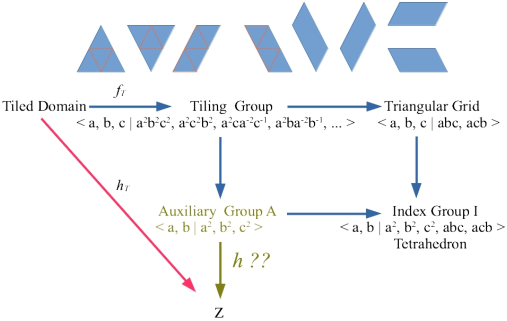
From Group Function to Height Functions for Lozenges



Remark : the index I ensures that $h_T(v) = h_{T'}(v) \mod 3$

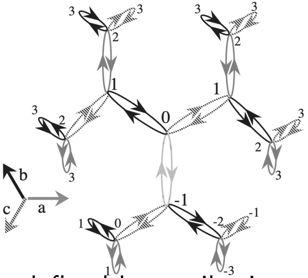
Group Function for Leaning Dominoes and Triangles

A new set of tiles, each of them covering four cells of the triangular grid



Order on A. From the Group Value to the Height Value

The presentation of the auxiliary group $A = < a, b, c | a^2, b^2, c^2 >$ is an infinite regular tree of degree 3. It induces a distance d_A on A

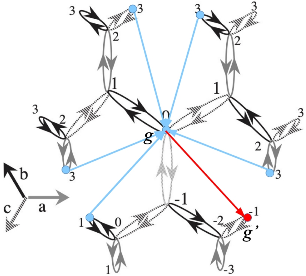


A partial order can be defined by attributing to each vertex a unique predecessor (and, therefore, two successors).
The height function is naturally defined by :

- $h(1_A) = 0$,
- If g' is the predecessor of g, then $h(g') - 1 = h(g)$.

Order on elements of A of same index

Two elements g, g' of A are *neighbors* if there exists x, y, z such that $\{x, y, z\} = \{a, b, c\}$ and $g\,xyz = g'$.
If, g and g' are neighbors, then they have the same index.
There exists a unique neighbor g' of g such that $h(g') < h(g)$.
For each other neighbor g'' of g, $h(g'') > h(g)$.



This allows to define $\min(g, g')$ for each pair of elements of A with the same index.

Order on Tilings

Definition : $T \leq T'$ if for each $v \in D$, $g_T(v) \leq g_{T'}(v)$.

Proposition : There exists a unique tiling T" such that $g_{T''} = \min(g_T, g_{T'})$.

Proof : based on the **Characterization Theorem.**
Let $g : D \to A$. There exists a tiling T of D such that $g = g_T$ if and only if

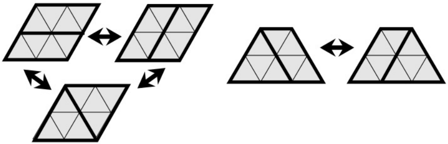
- $g(O) = 1_A$,
- for each $v \in D$, $index(g(v)) = index(v)$,
- for each edge (v, v') of D, $d_A(g(v), g(v')) \leq 3$.

Moreover, $g_T = g_{T'} \implies T = T'$.

A Convexity Lemma

Convexity Lemma : Let T be a tiling of a domain D and M. Assume that h_T has an interior local maximum in a vertex v_0. Then

- the tiling T is not minimal,
- a local flip, as below, can be done around v_0.



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Corollary : If T is minimal tiling, then h_T has no interior local maximum

Constructing the minimal Tiling

Proposition : Let T be a minimal tiling, and $M = \max\{h_T(v), v \in D\}$. Let v_0 such that $h_T(v_0) = M$, then

- v is on the boundary of D,
- T is completely determined in the neighborhood of v

Repeating this argument,

- the uniqueness of the minimal tiling is proved,
- an algorithm or tiling is exhibited,
- the flip connectivity is proved,

Results about Tilings with leaning dominoes and Triangles

- Flip Connectivity,
- Tiling algorithm,
  (in the same spirit of the one for dominoes)
- Computation of the number of flips between two tilings
  (not done in this lecture)

And without Triangles ?

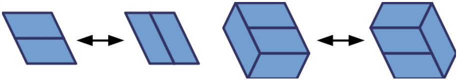
For tiling using only leaning dominoes, the group presentation

$$< a, b, c \mid a^2, b^2, c^2 >$$

and the induced height function can still be used.

Convexity Lemma : Let T be a tiling of a domain D. Assume that h_T has an interior local maximum in a vertex v_0 of G. Then,

- either a local flip, as below, can be done in T, which gives a tiling T such that $T < T'$,



- or each local minimum is contained in *zigzag*. (details on the next slide).

Zigzags

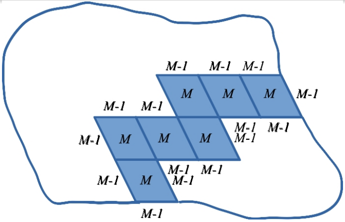


FIGURE – A zigzag.

Let T be a minimal tiling,

$$M = \max\{h_T(v), v \in D\} \qquad M' = \max\{h_T(v), v \in \delta D\}$$

Corollary : One of the following alternatives holds.

- either $M = M'$,
- or $M' = M - 1$, and all maxima of h_T are enclosed in zigzags

Detection of Zigzags

The group representation

$$< a, b, c \mid a, b >$$

allows to detect highest zigzags directed by a and b.