NOTES FOR CIMPA COURSE.
DYNAMICAL SYSTEM

NICOLAS BÉDARIDE

1. What is a dynamical system ?

Three different dynamical systems :

1.1. **Differential equation.** Let $x : \mathbb{R} \to \mathbb{R}^d$ be a function defined on an interval, let us denote $\partial_t x$ the derivative up to t. Then a differential equation of first order is

$$\partial_t x = f(x(t), t)$$

where f is a function from an open set Ω of $\mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}^d$.

An equation of the form $\partial_t x = f(x(t))$ is called an autonomus .

The map f is called a vector field because $(1, f(x(t))$ is the tangent vector to the trajectory $t \mapsto x(t)$.

Example 1.1. Example of Lorentz. Here $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where $x, y, z : \mathbb{R} \to \mathbb{R}$ with three positives coefficients σ, r, b

$$\begin{cases} \partial_t x = \sigma(y - x) \\ \partial_t y = rx - y - xz \\ \partial_t z = xy - bz \end{cases}$$

A differential equation of degree two is of the form $\partial_t^2 x = f(x, t)$. It can be reduced to a first order differential equation.

Example 1.2. Newton equation :

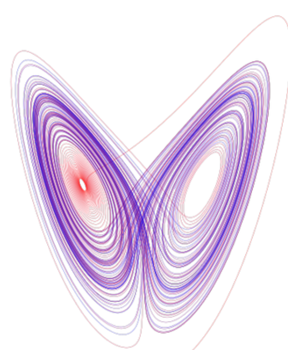


Figure 1. Lorentz

$$m\partial_t^2 x = F(x)$$

If we denote $p = m\partial_t x$, then $\partial_t p = F(x)$. Let us consider the function $H = \frac{1}{2m}p^2 + V$, with $F = -gradV$. We obtain $\begin{cases} \partial_t x = \frac{\partial H}{\partial p} \\ \partial_t p = -\frac{\partial H}{\partial x} \end{cases}$

A function f is locally lipschitz if for every $R > 0$, there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leqslant M|x - y| \quad |x|, |y| \leqslant R$$

Proposition 1.3. *A continuously differentiable function is locally lipschitz.*

Theorem 1.4 (Cauchy Lipschitz)**.** *If f is locally lipschitz on the second variable, then there is an unique function x defined on an interval containing 0 solution of*

$$\partial_t x = f(t, x), x(0) = x_0$$

Theorem 1.5. *If f is locally lipschitz, then the solution x is defined on a maximal interval (T_-, T_+). If $T_- \neq -\infty$, then $\lim_{T_-} |x(t)| = \infty$. Same thing for T_+.*

Example 1.6. Consider the Lorentz system with $r < 1$. We will prove that the solution exist for all $t > 0$. Indeed let us consider $V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$. Then we obtain $\partial_t V = 2\sigma[br^2 - (rx^2 + y^2 + b(z - r)^2)]$. Then consider $C > 0$ big enough such that the ellipsoid $V < C$ contains the ellipsoid $rx^2 + y^2 + b(z - r)^2 \leqslant br^2$, then solutions cannot escape from the ellipsoid, thus are bounded and thus exist for all $t > 0$.

Remark that it does not exclude the case where solution blow up in time $t < 0$.

Now let us introduce the notion of **phase space**.

Let $x(t, x_0)$ the solution of the Cauchy problem defined on interval $(T_-(x_0), T_+(x_0))$. Consider for $t \in \mathbb{R}$ the set $U_t = \{x_0 \mid T_- < t < T_+(x_0)\}$ and the map $\Phi_t : U_t \to \mathbb{R}^d$ such that $\Phi_t(x_0) = x(t, x_0)$. This map sends the initial data to the solution at time t.

We have $\Phi_t \circ \Phi_s = \Phi_{t+s}$

Example 1.7. Consider the motion of a pendulum of length l in the space with acceleration g : If we denote θ the angle of the pendulum with the vertical we have

$$\partial_{t,t}^2 \theta + \frac{g}{l} \sin\theta = 0$$

Consider $v = \partial_t \theta$, then we obtain

$$\begin{cases} \partial_t \theta = v \\ \partial_t v = -\frac{g}{l} \sin\theta \end{cases}$$

The phase space is $\mathbb{T} \times \mathbb{R}$.

Let $f() : \Omega \to \mathbb{R}^d$ be a vector field and let $x_0 \in \Omega$.

Consider Σ an open set of an affine hyperplane P, which contains x_0 and such that $\mathbb{R}^d = P \oplus \mathbb{R}f(x_0)$. We say that Σ is transverse to the orbit of x_0 under the flow. Then the time of first return of x_0 is given by $\Phi_{\tau_0}(x_0)$ where

$$\tau_0 = \min\{t > 0 \mid \Phi_t(x_0) \in \Sigma\},$$

Theorem 1.8. *If $\tau(x_0)$ exists, then*

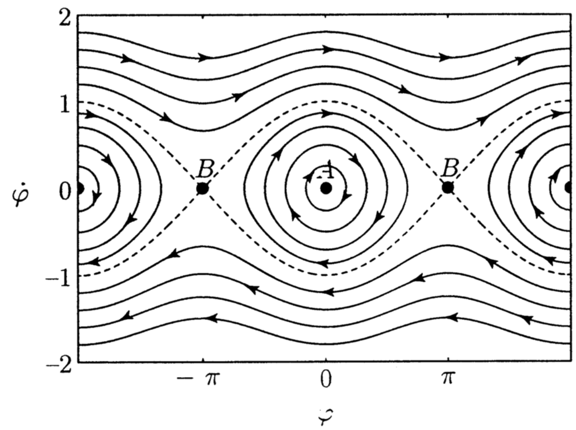


FIGURE 2. Phase space for the pendulum

— *There exists W open neighborhood of x_0 in Σ and a map $\tau : W \to \mathbb{R}$ such that for all $u \in W$, $\tau(u)$ is the first return time of u to Σ.*
— *the map $u \mapsto T(u) = \Phi_{\tau(u)}(u)$ is a diffeomorphism from W to its image.*

This theorem shows that the study of a differentia equation can be resumed to the study of some diffeomorphism T.

1.2. **Discrete dynamical system.** A discrete dynamical system is a map $f : X \to X$, and with $a \in X$ we study the sequences $x_{n+1} = f(x_n), x_0 = a$.

Remark the similitude with a differential equation : We replace x' by $x_{n+1} - x_n$.

The following equation is easy to solve

$$\partial_t x = \mu x(1 - x)$$

Consider the dynamics of population :

$$x_{n+1} = \mu x_n(1 - x_n)$$

Hard to understand :
— If $0 \leqslant \mu \leqslant 1$, then $\lim x_n = 0$
— If $1 \leqslant \mu \leqslant 3$, then $\lim x_n = \frac{\mu-1}{\mu}$. Two cases among if $\mu \leqslant 2$ or not.
— If $4 > \mu > 3$, a lot of adherence points. . .
— If $\mu > 4$, then $[0, 1]$ is not stable.

Another example is given by the following result : consider the Charkovski order (1964)

$$1 < 3 < 5 < \cdots < 2n+1 < \cdots < 2*3 < 2*5 < \cdots < 2^n*3 < 2^n*5 < \dots 2^n < 2^{n-1} < \cdots < 1$$

Theorem 1.9. *Let f a continuous map from $[0, 1]$ to $[0, 1]$. Consider the discrete dynamical system $([0, 1], f)$. If a point has period p then it has period q for all $q > p$ for the previous order.*

1.3. **Group action.** Let X be a compact space and G be a group which acts on X.

$$(g, x) \mapsto g.x$$

with the properties $h.(g.x) = (hg).x, e.x = x$.

We can look at the orbit of x : this is the set of $g.x$ with $g \in G$. One example is a $\mathbb{Z}$ action if we have a space X and an invertible map T : The action of $\mathbb{Z}$ is defined by $n.x = T^n x$.

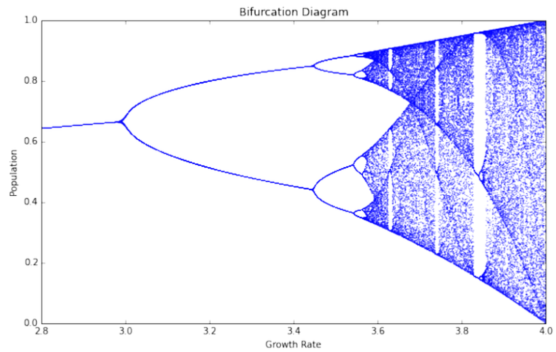


FIGURE 3. Bifurcation diagramm

If we want to look at a $\mathbb{Z}^2$ action, then we need two maps T, F which commute.

$$(n,m).x = T^n \circ F^m x$$

We can thing of it as a tiling of $\mathbb{Z}^2$ and the action of the group of translations $\mathbb{Z}^2$.

2. Ergodic theory

2.1. **Measured systems.** A **measurable dynamical system** is (X, T, μ) where X is a compact set, $T : X \to X$, and μ a probability measure on X such that $\mu(A) = \mu(T^{-1}A)$ for all measurable set $A \subset X$. We say that the measure μ is invariant with respect to (X, T).

Example 2.1. $x \to x+a$ on $\mathbb{T}$ with Lebesgue measure. $x \to 2x$ on $\mathbb{T}$ with Lebesgue measure. $x \to \varphi x$ on $\mathbb{T}$, see exercice for the invariant measure.

The system (X, T, μ) is **ergodic** if $T^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$. If there exists only one ergodic measure, then (X, T) is said to be **uniquely ergodic**.

Proposition 2.2. *Consider $f : X \to \mathbb{R}$. The system is ergodic if $f \circ T = f$ implies that f is constant almost everywhere.*

Proposition 2.3. *The set of invariant probability measures is a convex compact set. The ergodic measures are the extremal points of this set.*

Theorem 2.4 (Birkhoff)**.**
- *Let B be a measurable set, then the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\chi_B(T^i x)$ converges almost everywhere.*
- *If $f : X \to \mathbb{R}$ is in $L^1(X, \mu)$ then the sequence $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ converges almost everywhere to an invariant function $\overline{f}$.*

Corollary 2.5. *The system is ergodic if and only if for every $f \in L^1(X, \mu)$ the sequence converges to $\int_X f d\mu$ almost everywhere.*

Consider the **Koopman operator**

$$\begin{array}{ccc} L^2(X,\mu) & \to & L^2(X,\mu) \\ f & \mapsto & U_T(f) = f \circ T \end{array}$$

Proposition 2.6. *Let P be the orthogonal projection on the space of invariant vectors of U_T. Then for every f in $L^2(X, \mu)$ we have*

$$\lim \frac{1}{n}\sum_{i=0}^{n} U_T^i f = P(f)$$

Proposition 2.7. *The system is ergodic if and only if 1 is the only eigenvalue of U_T in $L^2(X, \mu)$.*

2.2. **Orbits.** The orbit of $x \in X$ is the sequence $O(x) = (T^n x)_{n\in\mathbb{N}}$.

Lemma 2.8 (Poincaré)**.** *Let (X, T, μ) be a dynamical system, and $A \subset X$ such that $\mu(A) > 0$. Then there exists $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$.*

Corollary 2.9. *Almost every point of A comes back to A infinitely often.*

The system is **minimal** if every orbit is dense in X. The system is **transitive** if one orbit is dense. One point is **periodic** if there exists $p \in \mathbb{N}$ such that $T^p x = x$.

2.3. **Spectral theory.** A function $f \in L^2(X, \mu)$ is an eigenfunction if there exists a complex number λ such that $Uf = \lambda f$. The set of eigenvalues form a countable subgroup of S^1. Now consider $V = \overline{< \text{eigenfunctions} >}$, this is the **spectrum**.

— If $V = L^2$, then we speak of **discrete** spectrum.
— If $V = \{\text{Const}\}$ we speak of **continuous** spectrum.
— If not, then we speak of **mixed** spectrum.

A system (X, T, μ) is **mixing** if it has non constant eigenfunctions. A system is **weak mixing** if every eigenfunction is constant almost everywhere.

Theorem 2.10. *A map is mixing if for all $A, B \subset X$ measurable sets, we have*

$$\lim_{n \to +\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

A map is weak mixing if for all $A, B \subset X$ measurable sets, we have

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0.$$

Proposition 2.11. *A mixing map is weak mixing, and a weak mixing map is ergodic.*

Theorem 2.12. *A map is weakly mixing if and only if $T * T$ is ergodic for the product measure.*

2.4. **Coding.** A *measured topological dynamical system* is a triple (X, T, μ) such that X is a compact topological space, μ is a finite measure defined on the Borel sets of X, and $T : X \to X$ is a μ-almost everywhere continuous map such that $\mu(T^{-1}(B)) = \mu(B)$ for any Borel set B of X.

To a measurable partition $(P_i)_{i \in I}$ of X, we associate its *coding* $cod : X \to I^{\mathbb{N}}$ defined by $cod(y) = (i_n)_{n \in \mathbb{N}}$ and $\forall n \in \mathbb{N}$, $T^n y \in P_{i_n}$. The map *cod* is a *symbolic coding* of the system (X, T) and the closure of $cod(X)$ defines a subshift over the alphabet I. A *generating partition* of the map T is a partition whose coding is injective almost everywhere.

Definition 2.13. A generating partition $(P_i)_{i \in I}$ of X is *regular* if every set $\overline{P_i}$ is the closure of its interior and if the boundary of each P_i is of zero measure.

2.5. **Examples.** We finish by four examples of different types which will be studied in all the following.

— A **subshift** defined by

$$\begin{array}{ccc} X & \to & X \\ x & \mapsto & Sx \end{array}$$

where $X \subset \mathcal{A}^{\mathbb{N}}$.

— **Rotations** on the torus $\mathbb{T}^d$.

$$\begin{array}{ccc} \mathbb{T}^d & \to & \mathbb{T}^d \\ x & \mapsto & x + a \end{array}$$

— Matrix action on $\mathbb{T}^d$, with $A \in SL_d(\mathbb{Z})$

$$\begin{array}{ccc} \mathbb{T}^d & \to & \mathbb{T}^d \\ x & \mapsto & Ax \end{array}$$

— A non compact example with link with Julia and Mandelbrot sets

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & z^2 + c \end{array}$$

3. Subshifts

3.1. **Definitions.** Let $\mathcal{A}$ be a finite set of cardinality d, and $\mathcal{A}^{\mathbb{N}}$ the set of infinite sequences. This set has the natural product topology, and is compact. This topology is metrizable with $d(u,v) = 2^{-n}$, where $n = \inf\{k \in \mathbb{N} | u_k \neq v_k\}$. Then we consider the shift map S on this set.

$$S((u_n)_{n\in\mathbb{N}}) = (u_{n+1})_{n\in\mathbb{N}}$$

A **subshift** is (X,S) where X is a subset of $\mathcal{A}^{\mathbb{N}}$ which is closed and S-invariant. A special case of subshift is the orbit of a point u :

$$X_u = \overline{\{S^n u, n \in \mathbb{N}\}}$$

Consider again a subshift X defined over the alphabet $\mathcal{A}$. Consider $x \in X$. A **word** of x of length k is a finite sequence $x_n \dots x_{n+k-1}$. The set of words of length n which appear in some $x \in X$ is called the **language** of the words of length n of X. It is denoted $\mathcal{L}_n(X)$, and the union of these sets is $\mathcal{L}(X)$ the **language** of the dynamical system. The language is factoriel : If uv belongs to it, then also do u and v. Let $v \in \mathcal{L}(X)$, then **the cylinder** defined by v is the set of elements of X which begin by v.

if $X = \mathcal{A}^{\mathbb{N}}$, then it is called **full shift**.

3.2. **Properties.**

Lemma 3.1. *With these notations*

- *The space $(\mathcal{A}^{\mathbb{N}}, d)$ is compact and complete.*
- *It is a Cantor space : no isolated point, totally disconnected and compact.*
- *The cylinder sets form a basis of the topology.*
- *The shift map is uniformely continuous on this space.*

Proposition 3.2. *The following points are equivalent*

- *The infinite word v is in X_u.*
- *For every integer n we have $L_n(v) \subset L_n(u)$.*
- *There exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that $v_0 \dots v_n = u_{k_n} \dots u_{k_n+n}$.*

Proposition 3.3. *We have equivalence between the points :*

- *A subshift X is transitive*
- *For every open sets $U, V \subset X$ there exists $x \in X$ and $n \in \mathbb{Z}$ such that $x \in U$ and $S^n(x) \in V$. We can also write it as*

$$\exists n \in \mathbb{Z}, S^{-n}V \cap U \neq \emptyset.$$

- *For every finite words u, v of the language, there exists $x \in X$ such that u, v belong to the language of x.*

Remark that the integer can be non positive.

Proposition 3.4.

- *A subshift X is minimal if and only if it does not contain a non empty subshift strictly included in X.*
- *The subshift is minimal if and only if for all $x, y \in X$ the languages of x and y are equal.*
- *Every subshift contains a minimal subshift.*

An infinite word is **ultimately periodic** if there exists an integer k such that $x_k x_{k+1} \dots$ is a periodic word.

Proposition 3.5. *The following points are equivalent.*
- *The element x is ultimately periodic.*
- *$O(x)$ is closed.*
- *$O(x)$ is finite.*

3.3. **Recurrence.** We give some important definitions about recurrence
- A sequence x is said to be **recurrent** if every word u of the language $\mathcal{L}_x$ appears infinitely many often.
- The sequence is said to be **uniformely recurrent** if for every n there exists N such that for every word $u \in \mathcal{L}_n(x)$ the size of the return word beteween two occurences of u is bounded by N.
- The infinite word x is said to be **linearly recurrent** if it is uniformely recurrent and there exists $k \geqslant 1$ such that $N \leqslant kn$ with previous notations.
- The subshift X is ?✠♠ if there exists $x \in X$ with property ?✠♠ and such that $X_x = X$.

Proposition 3.6. *We have implications* $1 \to 2 \to 3 \to 4$
(1) x is periodic,
(2) x is linearly recurrent,
(3) x is uniformely recurrent,
(4) x is recurrent.

Remark 3.7. No implication in other directions, see following examples :
- An sturmian subshift is quasi-periodic and non periodic.
- A sturmian subshift is LR if and only if its angle is with bounded partial quotient.
- There exists some substitution with a subshift uniformely recurrent and not LR.

Theorem 3.8 (Gottschalk)**.** *The subshift X_x is minimal if and only if x is uniformely recurrent.*

3.4. **Complexity function.** The **complexity function** of the language is the function

$$\begin{array}{ccc} \mathbb{N}^* & \to & \mathbb{N} \\ n & \to & p(n) = card\mathcal{L}_n(X) \end{array}$$

Proposition 3.9. *If x is an ultimately periodic word, then $p(n)$ is a bounded sequence.*
If there exists some integer n such that $p(n) \leqslant n$, then the sequence is ultimately periodic.

Lemma 3.10. *For every ✠language and every integers n, m we have*

$$p(n+m) \leqslant p(n)p(m).$$

Consider a language and the set L_n of words of length n of this language. A word of L_n is said to be **right special** if it admits several right expansions in a word of L_{n+1}. By the same way we define a **left special word**. A word is **bispecial** if it is right and left special. We denote $s(n) = p(n+1) - p(n)$ for every integer n.

An infinite word is a **sturmian word** if the complexity of this word equals $n+1$ for every integer n. A substitution is a **sturmian substitution** if the image of every sturmian word is a sturmian word.

Example 3.11. We will see that the fixed point of the Fibonacci substitution is a sturmian word.

Thus we can define the next notion. **The topological entropy** of the subshift is defined as

$$h(X) = \lim_{+\infty} \frac{\log p(n)}{n}$$

where p is the complexity function of the language.

4. Shifts of finite type

Here we consider sequences over $\mathbb{Z}$.

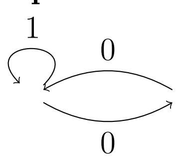
4.1. **Definitions.** Let F be a finite set of words over $\mathcal{A}$. A **subshift of finite type** is the set of infinite words $x \in \mathcal{A}^{\mathbb{Z}}$ such that no word of $\mathcal{L}(x)$ belongs to F. The set F is called the set of **forbidden words**.

Example 4.1. Consider $F = \{00, 01, 10\}$ for the alphabet $\{0, 1\}$.

Proposition 4.2. *The question to know if $\mathcal{A}^{\mathbb{Z}}$ is a SFT is decidable.*

Consider a finite graph, where edges are labelled by a finite set $\mathcal{A}$. It is called an **ω-automata**. We can associate a subshift such that the sequence $(e_n)_{n\in\mathbb{N}}$ belongs to the subshift if for every integer n we have $t(e_n) = i(e_{n+1})$ where $t(e)$ is the terminal vertex of the edge, and $i(e)$ is the initial one. We call this subshift a **sofic subshift** defined by the ω-automata.

Example 4.3. Example of sofic shift.



It is called **even shift**.

Proposition 4.4. *Every SFT is obtained from a ω-automata.*

Proposition 4.5. *If for every x, the sequence of edges defines an unique sequence of vertices, then the sofic subshift is a SFT.*

Remark is not the case of the even shift.

Proposition 4.6. *For every sofic subshift, there exists a SFT which projects onto it.*

Example 4.7. For the even shift we find $F = \{ab, bb, ca, cc\}$, and X_F is a SFT on a 3 letters alphabet $\{a, b, c\}$. Moreover we project b, c on 0 and a onto 1.

Thus a sofic subshift is a **factor** of a SFT.

Proposition 4.8. *Consider F a finite set of words and $X_F \subset \mathcal{A}^{\mathbb{Z}}$ the SFT associated. Assume that $X_{\mathcal{F}} \neq \emptyset$ then it contains one periodic word.*

4.2. **SFT and graph map.** Consider an oriented graph. Let $A \in \mathcal{M}_N(\mathbb{N})$ be a matrix which will be the adjacy matrix of the graph. The coefficient $A_{i,j}$ is 1 if there is an edge from vertex i to vertex j.

To a SFT we can associate a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs, given by (V_n, E_n) where :

— V_n is the set of words of length n in the language.

— E_n represents the words of length $n + 1$,

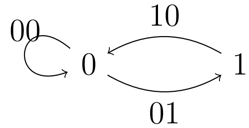
— The edge between vertices U, V exists if one can find a, b such that $Ua = bV$. In this case we label it by Ua.

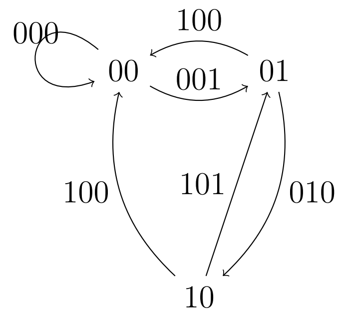
Lemma 4.9. *The study of the SFT is the same as the study of G_n where $n + 1$ is the maximum size of the forbidden words.*

Démonstration. ✠Vérifier. ✠ □

Remark 4.10. To a sofic subshift we can associate the same graph. But in this case several edges can have the same label.

Example 4.11. Consider $F = \{11\}$ we obtain the following graphs and the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.





We are not obliged to label the edges.

Remark that the matrix is irreducible if and only if the graph is strongly connected.

Now we define the notion of **sliding block code**. Consider two integers m, n and a function Φ defined on an interval centered on i and translated of $[-m, n]$ by

$$\Phi(x_{i-m} \dots x_{i+n}) = y_i$$

Example 4.12. The following map allows us to pass from the golden mean shift to the even shift

$$\begin{cases} \Phi(00) = 1, \\ 0 = \Phi(01) = \Phi(10) \end{cases}$$

Remark the following fact, at the base of the theory of **cellular automata**

Proposition 4.13. *A sliding block code commutes with the shift maps and is continuous.*

4.3. **Classical examples.**

Example 4.14. We consider three examples given by the set of forbidden words over a two letter alphabet :

— $F = \{11\}$: **Golden mean shift**
— Range of 0 of odd length : **Even Shift**
— Range of 0 of the same length : Not a SFT, not a sofic susbhift.

Proposition 4.15. *The even shift is not an SFT.*

4.4. **Dynamics.** Recall that the topological entropy of a SFT is given by

$$h(X) = \lim_n \frac{\ln p(n)}{n}$$

where $n \to p(n)$ is the complexity function of the subshift.

Example 4.16. Consider the SFT given by $F = \{11\}$. The topological entropy is $\ln \varphi$.

Theorem 4.17. *The topological entropy of a SFT is equal to* $\ln \lambda$ *where* λ *is the spectral radius of the matrix.*

Proposition 4.18. *Let* $b(n)$ *be the number of periodic words of period exactly* d *in a subshift. For SFT we have* $\sum_{d|n} b(d) = tr(A^n)$ *for every integer* n.

Proposition 4.19. *For the complexity function we have*

$$p(n) = \sum_{i,j} M_{i,j}^{n-m}$$

where m is the length of the maximal forbidden word.

Example for the golden mean shift : $M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Remark that for prime number we have $b(p) = tr(M^p) - tr(M)$.

Remark 4.20. Remark that $\frac{\ln b(n)}{n}$ does not converge, even for a SFT.

A subshift is **topologically weak mixing** if for every words u, v of $\mathcal{L}_X$, there exists N such that for every integer $n \geqslant N$ there exists a word w of length n such that uwv is in the language $\mathcal{L}_X$.

Proposition 4.21. *A SFT is topologically weak mixing if and only if its matrix is primitive.*

5. Entropy

5.1. **Metric entropy.** Consider μ an ergodic measure of (X, T). Then for almost every x the following limit exists and is constant

$$\lim \frac{-1}{n} \log \mu([x_0 \dots x_{n-1}])$$

It is the **metric entropy** of the measure μ. We denote it $h_\mu(X, T)$.

Consider an invariant measure μ : By definition, it is the convex hull of a finite ✠number of ergodic measures : $\mu = \sum a_i \mu_i$. Then we define its metric entropy as

$$h_\mu = \sum a_i h_{\mu_i}$$

Theorem 5.1. *Consider a compact dynamical system*

$$h_{top}(X) = \sup_\mu h_\mu(X)$$

5.1.1. *Parry measure for a SFT.*

Definition 5.2. The **Parry measure** is an ergodic measure μ which maximizes the topological entropy of the SFT.

$$h_\mu = h_{top}$$

Proposition 5.3. *If the SFT is topologically transitive, then the Parry measure is unique. Moreover we can compute it by the following formula. Let us denote*

$$p_i = l_i r_i, p_{i,j} = M_{i,j} \frac{r_j}{\lambda r_i}$$

where r, l are right and left eigenvectors of the incidence matrix for the Perron Frobenius eigenvalue λ. The measure of a cylinder $[v]$ defined by the word $v = v_0 \dots v_n$ is given by

$$\mu([v]) = p_{v_0} \prod_{i=0}^{n-1} p_{v_i v_{i+1}}$$

For example we obtain the following examples

$$\mu([ab]) = p_a p_{a,b}, \mu([abc]) = p_a p_{ab} p_{bc}.$$

5.2. **Formal definition of the metric entropy.** Consider a partition Q of X, then we define $T^{-1}Q$ as the union of $T^{-1}Q_i, i = 1 \dots k$. For two partitions P, Q we define $P \vee Q$ as the refinement of the two partitions

$$\{Q_i \cap R_j, \mu(Q_i \cap R_j) > 0\}$$

Then we consider $\bigvee_{i=0}^{N} T^{-i}Q$.

The entropy of the partition is defined as $H(Q) = -\sum \mu(Q_i) \log \mu(Q_i)$.

The measure entropy of the system with respect to Q is then defined as

$$h(T, Q) = \lim_{+\infty} \frac{1}{N} H(\bigvee_{i=0}^{N} T^{-i}Q)$$

Then the metric entropy is

$$h_\mu(T) = \sup_Q h(T, Q).$$

Theorem 5.4 (Sinai)**.** *In the previous formula the supremum is obtained for partitions which are generators.*

Recall that a partition is generator if μ almost every point has an unique symbolic name.

6. Substitutions

6.1. **Definitions.** Consider a finite set $\mathcal{A}$, then denote $\mathcal{A}^*$ the set of finite words defined over $\mathcal{A}$. A **substitution** is a morphism σ of this monoid onto itself.

$$\sigma(uv) = \sigma(u)\sigma(v)$$

Fix a basis $(e_1 \dots e_d)$ of $\mathbb{R}^d$. There exists a map π from $\mathcal{A}^*$ into $\mathbb{Z}^d$ where d is the cardinality of $\mathcal{A}$ given by :

$$\pi(w_0 \dots w_n) = \sum_{k=0}^{n} e_{w_k}.$$

This allows to define a linear morphism of $\mathbb{Z}^d$ which commutes with π, σ : The morphism of $\mathbb{Z}^d$ can be defined by a matrix M_σ, called the **incidence matrix** of the substitution.

The substitution is said to be :

— **primitive** if there exists an integer k such that $M_\sigma^k > 0$.

— **irreducible** if the characteristic polynomial of M_σ is irreducible over $\mathbb{Z}$.

— **unimodular** if $det(M_\sigma) = \pm 1$.

— **Pisot** if the dominant eigenvalue is a Pisot number.

We recall that a **Pisot number** is an algebraic number such that all its algebraic conjugate are in the unit disc.

The substitution acts on $\mathcal{A}^*$ and it can be extended to an action on $\mathcal{A}^{\mathbb{N}}$.

A **fixed point** of σ is an element of $\mathcal{A}^{\mathbb{N}}$ such that $\sigma(u) = u$. A **periodic point** is an element such that $\sigma^k(u) = u$ for some $k > 0$.

The **language** of a substitution is the set of finite words which appear as a subword of some $\sigma^n(a)$ where $a \in \mathcal{A}$. The **subshift** associated to a substitution is the set of sequences such that every subword appears in the language of σ. It is denoted X_σ.

A susbtitution is said to be **aperiodic** if the subshift is not made of periodic words.

6.1.1. *Automaton.* An **automaton** is a 5-uplet $(Q, \Sigma, \delta, q_0, F)$ where

— Q is a finite set of states.

— Σ is a finite set of symbols, called the alphabet.

— δ is a function $Q \times \Sigma \to Q$, called the transition function.

— $q_0 \in Q$ is the start state.

— F is the set of states, called the accept states.

An automaton reads a finite word $w = a_1 \dots a_n$ with $a_i \in \Sigma$ and a **run** of the automaton is a sequence of states $q_0 \dots q_n$ such that $q_i = \delta(q_{i-1}, a_i)$ for $0 < i \leqslant n$. The word w is **accepted** if q_n belongs to F.

Let k be an integer greatest or equal than one. One special class is given by the **k-automaton**. It is a directed graph defined by

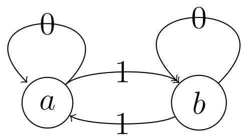
— A finite set of vertices called S, and one initial vertex called i.

— k oriented edges from S to S denoted $0 \dots k-1$.

— A set Y and a map ϕ from S to Y called the output function.

A sequence $(u_n)_{n\in\mathbb{N}}$ is called **k-automatic** if we write $n = \sum_{i=0}^{j} n_i k^i$ and starting from the initial state we follow a path in the oriented graph defined by $n_0, \dots n_j$. At this point we are at vertex $a(n)$ and we have $u_n = \phi(a(n))$.

Proposition 6.1. *The following automaton is linked to the Thue-Morse subshift. The initial state is a and the output is given by $Id_{\{a,b\}}$.*



Proposition 6.2. *The term u_n of the fixed point is equal to the sum of the digits mod 2 of the expansion of n in base 2.*

For example $(18)_2 = 10010$, thus $u_{18} = 0$.

Theorem 6.3 (Perron Frobenius)**.** *Consider a primitive matrix, then there exists $\lambda > 0$ which is eigenvalue of M with eigenspace of dimension one, and such that other eigenvalues θ fulfill $|\theta| < \lambda$. Moreover a basis of the eigenspace of M associated to λ has positive coefficients.*

Theorem 6.4 (Mossé)**.** *A substitution σ is aperiodic if and only if for every $u \in X_\sigma$, there exists a unique integer k and an unique $v \in X_\sigma$ such that $S^k\sigma(v) = u$.*

Automaton of prefixes-suffixes, see [?] and [?].
Consider an aperiodic substitution. Let $w \in X_\sigma$, then by previous theorem there exists an unique $v \in X_\sigma$ and an unique $k < |\sigma(v_0)|$ such that $w = S^k\sigma(v)$. We define a map

$$\theta : \begin{array}{ccc} X_\sigma & \to & X_\sigma \\ w & \mapsto & v \end{array}$$

Then consider

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A} | \exists b, \sigma(b) = p.a.s\}$$

Now define the application $\gamma : X_\sigma \to \mathcal{P}$ which sends w to (p, w_0, s) such that $\sigma(\theta(w)_0) = p.w_0s$. The sequence $\gamma(\theta^i(w))_{i\in\mathbb{N}}$ is called the **development in prefix-suffixes**. Then we define an automaton such that

— The set of states is $\mathcal{A}$.
— The set of edges is $\mathcal{P}$.
— There is an edge from a to b if $\sigma(b) = p.a.s$. The edge is labelled by (p, a, s).

6.2. **List of classical examples.**

$\begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}$	$\begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases}$	$\begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$	$\begin{cases} 0 \to 0010 \\ 1 \to 1 \end{cases}$
Fibonacci	**Thue – Morse**	**Tribonacci**	**Chacon**

6.3.

Lemma 6.5. *Assume that for each letter b, we have $\lim_{+\infty} |\sigma^n(b)| = +\infty$, then there is a periodic point.*

Proposition 6.6. *Let σ be a substitution such that : there exists a letter a with $\sigma(a)$ starting with a.The substitution is everywhere growing. Every letter appears in the fixed point starting with a. Then σ is primitive if and only if the fixed point beginning with a is minimal.*

6.4. Dynamics.

Theorem 6.7. *The subshift of a primitive substitution is minimal and uniquely ergodic.*

Proposition 6.8. *The* **Chacon** *substitution defines a minimal subshift :*

$$\begin{cases} 0 \mapsto 0010 \\ 1 \mapsto 1 \end{cases}$$

Proposition 6.9. *For a primitive substitution, the frequence of a letter i is given by*

$$\frac{R_i}{\sum R_j},$$

where R is a right eigenvector of M_σ associated to the Perron Frobenius eigenvalue.

6.5. Combinatorics on substitutions.

Theorem 6.10 (Pansiot)**.** *For every substitution σ, the subshift X_σ verifies : $p_X(n) \leqslant Cn^2$. If the substitution is primitive, then the complexity function is at most linear.*

Corollary 6.11. *For every substitution, the topological entropy of X_σ is zero.*

Proposition 6.12. *The Thue-Morse word, associated to the substitution θ fulfills*
- *The strong bispecial words are $\theta^n(ab), \theta^n(ba)$.*
- *The weak bispecial words are $\theta^n(aba), \theta^n(bab)$.*
- *The neutral bispecial words are a, b.*
- *The complexity function is sub-linear.*
- *It is an aperiodic word.*

Theorem 6.13 (Brleck)**.** *The complexity function of the Thue-Morse word is equal to*

$$p(n) = \begin{cases} 6.2^{r-1} + 4p, 0 \leqslant p \leqslant 2^{r-1} \\ 8.2^r + 2p, p > 2^{r-1} \end{cases}$$

where $n = 2^r + p + 1$.

Proposition 6.14. Cobham Theorem *Consider the set $E = \{2^n, n \in \mathbb{N}\}$. This set is 2-automatic.*

Démonstration. Consider the substitution $a \mapsto ab, b \mapsto bc, c \mapsto cc$ and the map ϕ given by $a, c \mapsto 0, b \mapsto 1$. Then let x the fixed point of the substitution which begins by a. We have $\mathbf{1}_E = \phi(x)$. □

7. Translations

7.1. Definitions. The **torus** is a topological space quotient of $\mathbb{R}^n$ by a lattice of rank n. It is a compact connected group. In mots of the case we will consider $\mathbb{R}^n/\mathbb{Z}^n$. We denote an element by $[x]$ where $x \in \mathbb{R}^n$. A **translation** is a map of the following form with $\alpha \in \mathbb{R}^n$.

$$\begin{array}{ccc} \mathbb{T}^n & \to & \mathbb{T}^n \\ [x] & \mapsto & [x+\alpha] \end{array}$$

The vector α is **totally irrational** if we have

$$a_1\alpha_1 + \cdots + a_n\alpha_n = b, a_i \in \mathbb{Q}, b \in \mathbb{Q} \Longrightarrow a_1 = \cdots = a_n = b = 0$$

7.2. Dynamical properties.

Proposition 7.1. *The Lebesgue measure is invariant by the translation*

Démonstration. Left to the reader □

Theorem 7.2. *A translation by α is uniquely ergodic if α is totally irrational.*

Proof for $n = 1$ later.

Proposition 7.3. *If α is a rational number then every point has a periodic orbit. If α is an irrational number then the translation is minimal.*

Démonstration. If $\alpha = \frac{p}{q}$, then it is an easy exercice. Indeed $T^q(x) = x + p \mod 1 = x$. Thus every point is periodic with a period which divides q.
Now consider the case α irrational number, and let $x \in \mathbb{T}^1$. All the points $x_k = x + \{k\alpha\}$ are distinct points. Let N be a positive integer and consider $\{k\alpha\}$ (fractional parts), $1 \leqslant k \leqslant N+1$. Now we consider the N intervals of length $1/N$ which make a partition of $[0,1]$. By the pigeon hole principle, 2 points are in the same interval. Let us denote them x_n, x_m and assume $m > n$. Let us denote b the distance between these 2 points : by definition it is less than $1/N$. Now consider the points x_{n+ik} with i positive integer and $k = m-n$. We compute $x_{n+(i+1)k} - x_{n+ik}$, and we remark that two consecutive points are at distance at most b. We deduce that every point of the circle is at distance at most b of one of them. We let N go to infinity, and we deduce that the orbit of x is dense. □

Remark the result is different in higher dimension.

Example 7.4. Consider the two following translations

$$\begin{array}{ccc} \mathbb{T}^2 & \to & \mathbb{T}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x + \frac{2}{3} \\ y + \sqrt{2} \end{pmatrix} \end{array} \quad \begin{array}{ccc} \mathbb{T}^2 & \to & \mathbb{T}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x + \sqrt{2} \\ y + \sqrt{2} \end{pmatrix} \end{array}$$

No one is a minimal system.

Proposition 7.5. *On $\mathbb{T}^1$, a translation of irrational vector is ergodic for the Lebesgue measure.*

Démonstration. We use a criteria with invariant function. Assume there exists a map $f \in L^2(X,\mu)$ such that $f \circ T = f$. We use Fourier decomposition of $f : f = \sum a_n e^{inx}$,and we obtain $a_n e^{in\alpha} = a_n$. Since α is an irrational number we deduce, $a_n = 0$ for every non zero integer n. Thus the map f is a constant function. □

The dynamical properties of translation can be summit it :

Theorem 7.6. *A translation*
- *A translation by a totally irrationnal vector is uniquely ergodic.*
- *A translation has discrete spectrum.*
- *It is not mixing or weak mixing*
- *An invertible, ergodic map with discrete spectrum is isomorphic to a translation on a compact group.*
- *Its maximal equicontinuous factor is itself.*

Remark that a periodic subshift also have a discrete spectrum.

7.3. Coding.

Definition 7.7. A finite partition $(P_i)_{i\in I}$ of P/Λ is said to be *liftable* with respect to the translation $T_x : y \mapsto y + x$ of P/Λ if there exists :
- a fundamental domain $D \subseteq P$ for the action of Λ
- a partition $(D_i)_{i\in I}$ of D
- some vectors $(t_i)_{i\in I}$ in P^I

such that for every i in I :
- $D_i + t_i \subseteq D$
- $\pi(D_i) = P_i$
- $\pi(t_i) = x$

where $\pi : P \to P/\Lambda$ is the quotient map.

Theorem 7.8 (Baryshnikov)**.** *On the torus $\mathbb{T}^d$ consider the coding of a minimal translation by polytopes, obtained by the billiard map. Then the subshift fulfills*

$$p(n) \leqslant Cn^d.$$

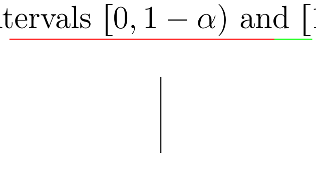
With more hypothesis we obtain an exact formula independant of the direction.

7.3.1. *Case of the torus $\mathbb{T}^1$.* Remark that $\mathbb{T}^1$ is isomorphic to S^1 and thus we can look at the following map which is a rotation.

$$\begin{array}{ccc} S^1 & \to S^1 & \\ z & \mapsto & ze^{2i\pi\alpha} \end{array}$$

Coding of the translation Consider $[0, 1)$ as a fundamental domain of the torus. Then a translation is an exchange of two intervals.

Let us denote $\alpha$the translation vector. We code the translation with the partitions in intervals $[0, 1-\alpha)$ and $[1-\alpha, 1)$.



The coding is thus given by

$$\begin{array}{ccc} \mathbb{T} & \to & \{0,1\}^{\mathbb{N}} \\ x & \mapsto & (u_n)_{\mathbb{N}} \end{array}$$

avec $u_n = \phi(R^n x)$ et $\phi(x) = \begin{cases} 0 & x \in [0, 1-\alpha[\\ 1 & \text{sinon} \end{cases}$

Proposition 7.9. *If α is irrational, then we obtain sturmian words, of complexity $p(n) = n+1$.*

If α is a rational number, then every point has a periodic orbit, thus $p_u(n) \leqslant C$.

With this proposition we know that there exists at least one sturmian word and we know a method to construct a lot of sturmian words.

Example 7.10. Consider the two following translations

$$\begin{array}{rcl} \mathbb{T}^1 & \to & \mathbb{T}^1 \\ x & \mapsto & x + \frac{2}{3} \end{array} \quad \begin{array}{rcl} \mathbb{T}^1 & \to & \mathbb{T}^1 \\ x & \mapsto & x + \frac{1}{\varphi} \end{array}$$

We can describe the language of the first one by hands. For the second one, we refer to one exercice.

7.3.2. *Case of the torus $\mathbb{T}^2$.*

Proposition 7.11. *Consider the euclidean torus with fundamental domain $[0,1]^2$. A translation of the torus $\mathbb{T}^2$ is an exchange of four rectangles.*

Proposition 7.12. *Consider the torus with an hexagon as fundamental domain. Then a translation of $\mathbb{T}^2$ is an exchange of three rhombi.*

7.3.3. *Example of Tribonacci fractal.*

Theorem 7.13 (Rauzy)**.** *On considère une racine complexe α de $X^3 - X^2 - X - 1$ et le tore $\mathbb{C}/(\mathbb{Z} + \alpha\mathbb{Z})$. Alors il existe un fractal du plan domaine fondamental de ce tore, tel que la translation $z \mapsto z + \alpha^2$ soit conjugué au subshift défini par la substitution $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$.*

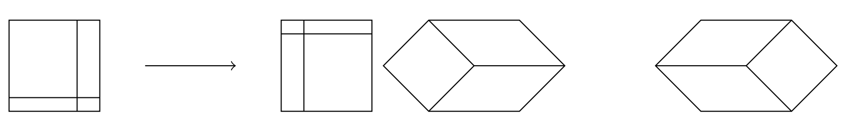


FIGURE 4. Échange de morceaux dans le tore $\mathbb{T}^2$.

8. Exercices

8.1. Subshifts.

Exercice 1. Consider the map $x \mapsto 2x \mod 1$ on $\mathbb{T}^1$. Prove that for every integer n the point $\frac{p}{2^n-1}, 0 \leqslant p < 2^n - 1$ is periodic of period n.

Exercice 2. Consider $X = \{(01)^\omega, (10)^\omega\}$. Prove it is a subshift and compute the complexity function of this subshift.

Exercice 3. Consider $X \subset \{0,1\}^\mathbb{Z}$ the set of sequences which contain exactly one 1. Show that X is shift-invariant, but that X is not a subshift.

Exercice 4.

(1) What is the closure of the orbit of $x = 01111\ldots$ under the shift map ?
(2) Is the following subshift minimal, transitive ? $X = \{0^\omega\}$.

Exercice 5. The subshift is irreducible if for every finite words $u, v \in \mathcal{L}(x)$, there exists $w \in \mathcal{L}(X)$ such that uwv also belongs to the language

(1) Find an example of an irreducible subshift.
(2) Find an example of a non irreducible subshift.

Exercice 6. Consider an alphabet with 2 letters, and X the set of sequences such that $u_n = 1$ implies $u_{n+1} = u_{n+2} = 0$.

(1) Find the elements of X fixed by the shift map.
(2) Prove that the frequency of 1 in an element of X is not well defined.

Exercice 7. Consider the translation by $\frac{3}{7}$ on $\mathbb{T}^1$.

(1) Describe the different orbits.
(2) Find two invariant measures.

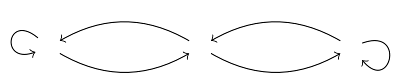
8.2. SFT.

Exercice 8. Describe the subshift of finite type defined by

$$F = \{00, 101\}.$$

Exercice 9. Compute the entropy of the even shift map.✠

Exercice 10. Show that the subshift described by the following graph is sofic : The two loops have the same name, and the other edges have different names.



Find a SFT which projects on it.

Exercice 11. Consider the following sliding block code on the full shift on a two letter alphabet :

$$\varphi(abcd) = b + a(c+1)d \mod 2$$

Consider its restriction to $[-1, 2]$. Compute the images of $1001, 1101$. What can you remark ?

Exercice 12. On a two letters alphabet, describe some properties of the SFT of zero entropy.

Exercice 13. Find the bispecial words of the language of the SFT given by $\mathcal{F} = \{11\}$.

Exercice 14. Compute the Parry measure for the SFT given by $F = \{111\}$.

8.3. Substitutions.

Exercice 15. Consider the following substitution

$$\begin{array}{rccc} \sigma : & \mathcal{A}^* & \to & \mathcal{A}^* \\ & a & \mapsto & aba \\ & b & \mapsto & a \end{array}$$

(1) Compute the incidence matrix, and show it is a primitive substitution.

(2) Compute the frequencies of the letters.

Exercice 16. Consider a factorial language. A Rauzy graph G_n is a graph where vertices are words of length n of the language, and there is an oriented arrow between u and v if there exist two letters a, b such that $ua = bv$ and ua belongs to the language.

(1) Draw the Rauzy Graph $G_n, n = 2, \dots 4$ for the Fibonacci word.

(2) Do the same thing for the Thue Morse word.

(3) What are the differences ?

Exercice 17. Compute the complexity of the language of the subshift defined by the substitution

$$\begin{array}{rccc} \sigma : & \mathcal{A}^* & \to & \mathcal{A}^* \\ & a & \mapsto & ab \\ & b & \mapsto & ac \\ & c & \mapsto & a \end{array}$$

Exercice 18.

(1) Find one substitution with a non minimal subshift.

(2) find a substitution, where every element of the subshift is periodic.

(3) Find a substitution over a three letters alphabet, where the frequencies of each letter is a rational number.

Exercice 19. For the Fibonacci substitution compute the measures of the cylinders [0] and [01].

8.4. Measures.

Exercice 20. Consider the map

$$T(x) = \begin{cases} x + \varphi - 1 & [0, 2 - \varphi) \\ x + \varphi - 2 & [2 - \varphi, 1) \end{cases}$$

— Prove that there is no periodic point.

— Is there a link with a rotation on $\mathbb{T}^1$?

— Consider the first return map S of T on $[2 - \varphi, 1)$

$$S(x) = T^k x, k = \inf\{n, T^n x \in [2 - \varphi)\}$$

Compute S. What is the link between S and T ?

— Consider the subshifts associated to T, S on a 2 letters aphabet. Compute the coding of $x \in [2 - \varphi, 1)$ for the two maps, and show that they are related by a substitution.

Exercice 21. Consider an ergodic measure μ of the system (X, T). Let A be a set such that $T^{-1}A \subset A$, then prove that $\mu(A)$ is equal to 0 or 1.

Exercice 22. Prove that the map $x \mapsto (x + \sqrt{2}) \mod 1$ defined on $\mathbb{T}^1$ is ergodic for the Lebesgue measure. Compute its spectrum.

Exercice 23. Consider the set $X = [0, 1]$.

(1) Prove that every element $x \in X$ can be written in an unique way as $x = \sum x_n/2^n$ with $x_n \in \{0, 1\}$ and x_n non ultimately equal to 1.

(2) Now we define $T : X \to X$ by $Tx = y$ with $\begin{cases} y_n = x_{n+2}, n = 2k+1 \\ y_2 = x_1 \\ y_n = x_{n-2}, n = 2k \end{cases}$. Prove that T preserves the Lebesgue measure.

(3) Prove that (X, T) is transitive.

Exercice 24. Consider the map $x \mapsto \frac{1}{x} \mod 1$. Prove that the following measure $d\mu = f dx$ is invariant : $f(x) = \frac{1}{\log 2} \frac{1}{1+x}$

Exercice 25. Consider the dynamical system defined over $[0, 1]$ by$x \mapsto 4x(1 - x)$. Prove that the following measure is an invariant measure

$$\mu(B) = \frac{1}{\pi} \int_B \frac{dx}{\sqrt{x(1-x)}}$$

Exercice 26. Consider the system $x \mapsto \varphi x \mod 1$ and let us denote $\alpha = \varphi^{-1}$. Prove that the measure $d\mu = h(x)dx$ is an invariant measure :

$$x \mapsto h(x) = \begin{cases} \frac{1}{\alpha+\alpha^3} & [0, \alpha] \\ \frac{\alpha}{\alpha+\alpha^3} & \end{cases}$$

Exercice 27. Consider the system with $X = [0, 1]$ and

$$x \mapsto Tx = \begin{cases} 2x & 0 \leqslant x \leqslant 1/2 \\ 2(1-x) & x > 1/2 \end{cases}$$

Prove that the Lebesgue measure is invariant and ergodic. To which subshift is related this map ?

Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France.
Email address: nicolas.bedaride@univ-amu.fr