Bratteli-Vershik Models of Cantor Minimal Systems

Maryam Hosseini

CIMPA School, Bejaia, Algeria

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Goal.

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X is a Cantor Set and T *is a minimal homeomorphism on it.*

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To "model" a minimal Cantor system with a sequence of tower partitions and morphisms.

We are going to learn about:

Bratteli Diagrams, Vershik systems, Kakutani-Rokhlin partitions, Strong Orbit equivalence (with the time permission).

Minimal Dynamical Syetms on Cantor Sets.

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Examples: Odometers, primitive substitution, interval exchange transformations,... They all can be considered as a Bratteli-Vershik system.

Let's see what a Bratteli-Vershik system is.

Bratteli Diagram.

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- The graph B has infinite number of vertexes that are grouped in an infinite sequence of finite sets:

$$V = V_0 \sqcup V_1 \sqcup V_2 \sqcup \cdots , \quad V_0 = \{v_0\}, \ |V_i| = n_i$$

So

$$V_i = \{v_1^{(i)}, v_2^{(i)}, \dots, v_{n_i}^{(i)}\}.$$

- The set of edges of the graph is:

$$E = E_0 \sqcup E_1 \sqcup E_2 \sqcup \cdots ,$$

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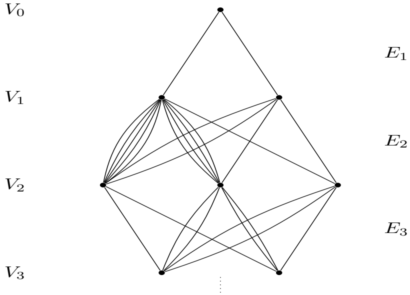
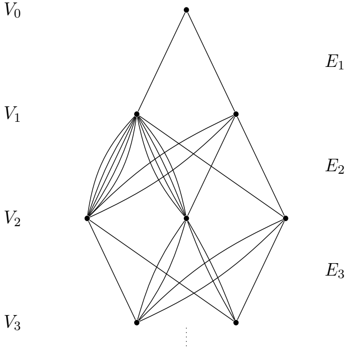


- the number of edges between $v_\ell^{(i)}$ and $v_k^{(i+1)}$ is equal to $M_i(\ell, k)$. So M_i is a $|V_{i+1}| \times |V_i|$ matrix.

$$M_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

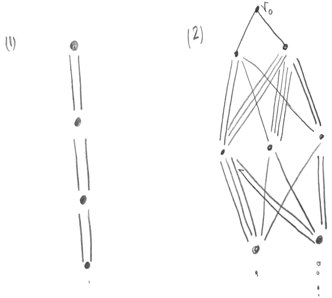
$$M_2 = \begin{bmatrix} 5 & 2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$



For diagram (1): all $M_i = [2]$ and for diagram (2):

$$M_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 1 & 2 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \dots$$



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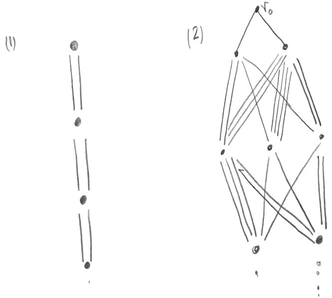

Each edge e has a source vertex: $s(e)$ and a range vertex: $r(e)$. In fact:

$$r : E_{i+1} \to V_{i+1}, \quad s : E_{i+1} \to V_i$$

Partial ordering on E.

For every $i \geq 1$ and every $v \in V_i$, consider the set:

$$E_v = \{e \in E_i : \quad r(e) = v\}$$

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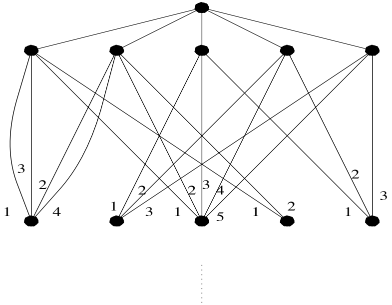
and make a lexicographic ordering on it.



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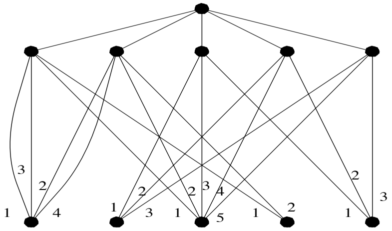

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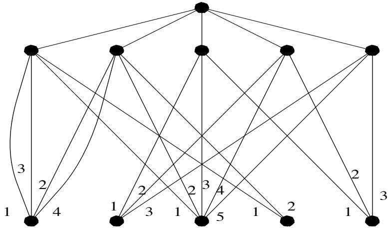

In this way for every i, E_i will be a partially ordered set. Indeed, Every two edges e, e' are comparable, $e \leq e'$, if and only if $r(e) = r(e')$.

Then we say that $B = (V, E, \leq)$ is an ordered Bratteli diagram.

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Remark. Since the ordering on $r^{-1}(v)$ is linear (lexicographic) we can consider the minimum ordinal number and the maximum ordinal number for it.

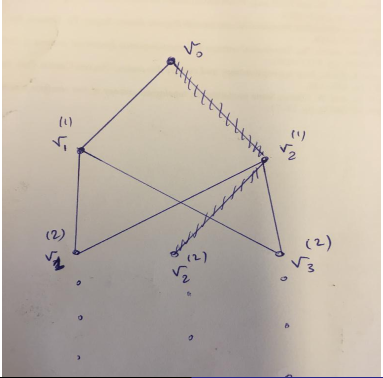
$$r^{-1}(v) = \{e_{\min}, \ldots, e_{\max}\}.$$

Finite paths.

For every $\ell, k \in \mathbb{N}$, $k < \ell$ consider the set of all (finite) paths between V_ℓ and V_k:

$$P_{\ell,k} = \{(e_{k+1}, e_2, \dots, e_\ell) : \ r(e_i) = s(e_{i+1}), s(e_{k+1}) \in V_k, r(e_\ell) \in V_\ell\}.$$

Below, there are 5 finite paths from level V_0 to level V_2.



Orderings on finite paths.

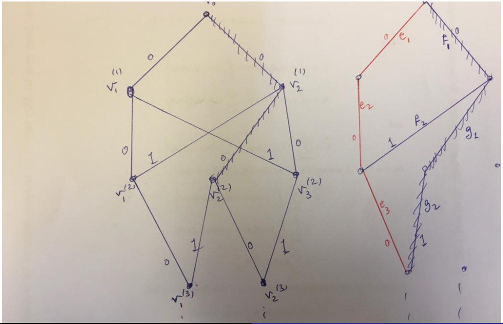
consider all finite paths from V_ℓ to a fixed vertex in V_k. In fact,

$$(e_{k+1}, e_2, \ldots, e_\ell) > (e'_{k+1}, e'_2, \ldots, e'_\ell)$$

if and only if

$$\exists i;\ \ k+1 \le i \le \ell;\ \ e_i > e'_i,\ \ \ \forall i < j \le \ell\ \ \ e_j = e'_j.$$

$$\text{Below} : (e_1, e_2, e_3) \le (f_1, f_2, e_3) \le (f_1, g_1, g_2).$$



Existence of infinite min and max paths.

Fist of all, by Konig's lemma, there are infinite paths on each Bratteli diagram.

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For every $v \in V_n \subset V$ there exists a unique finite path in $E_{\max}$ (resp. $E_{\min}$) from $v_0 \in V_0$ to v.

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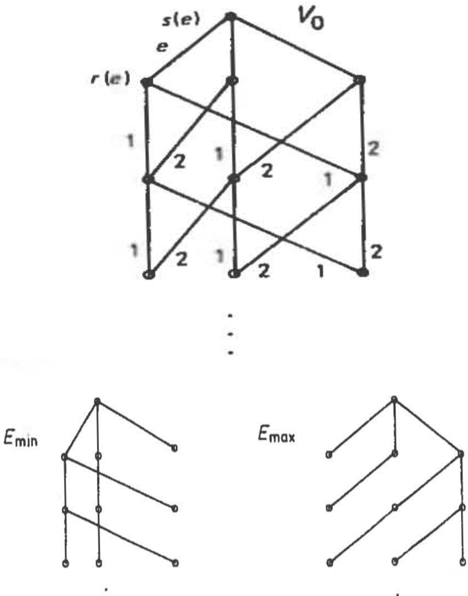
For every $v \in V_n \subset V$ there exists a unique finite path in $E_{\max}$ (resp. $E_{\min}$) from $v_0 \in V_0$ to v.

Let $E_{\max}$ (resp. $E_{\min}$) be the set of all maximal (resp. minimal) edges in the partially ordered set E and consider the subdiagram(s) containing only $E_{\max}$ (resp. $E_{\min}$).


s(e)
V_0
e
r(e)
1
2
1
2
1
2
1
2
1
2
1
2
E_{min}
E_{max}

...

Then by the Konig's lemma there exist infintie paths in these two subdiagrams.

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If we start from v_0, we can just pass through edges with minimal (resp. maximal) ordinal numbers to make infinite mini (resp. max) path(s).

The topological space X_B.

For a Bratteli diagram $B = (V, E)$ consider the set of all infinite path with the initial source $v_0 \in V_0$:

$$X_B = \{(e_1, e_2, \dots) : \quad e_1 \in E_1, \quad r(e_i) = s(e_{i+1})\}.$$

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Equipe X_B with the topology generated by the following cylinder sets:

$$U(e_1, e_2, \dots, e_k) = \{(f_1, f_2, \dots) \in X_B; \; f_i = e_i, 1 \leq i \leq k\}.$$

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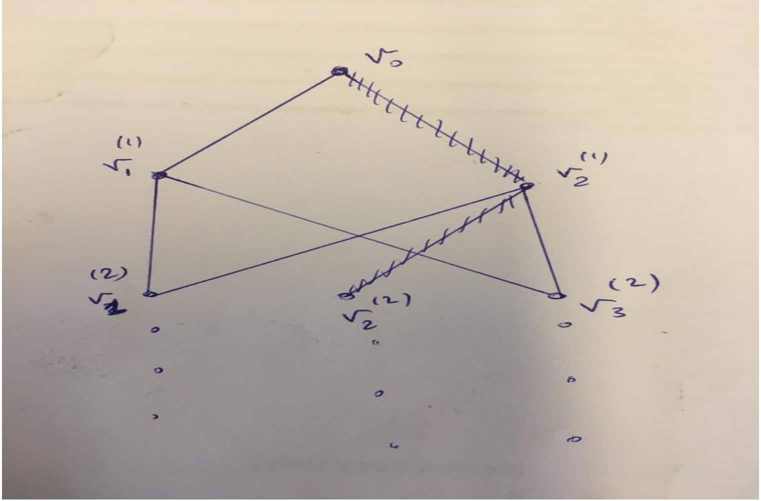
So we consider the set of all these cylinder sets as the basis of the topology on X_B.


V_0
$V_1^{(1)}$
$V_2^{(1)}$
$V_1^{(2)}$
$V_2^{(2)}$
$V_3^{(2)}$

- Cylinder sets are in fact finite paths from v_0.
- Each of the above cylinder sets is also closed. Because the number of finite paths from the top is finite and so the complement of each cylinder is a finite union of such cylinder sets. So they are clopen.

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- Remark. To define the topology we didn't need the ordering.

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$$d(x,y) < \delta \quad \Rightarrow \quad x, y \text{ agree on their first } k_0 \text{ edges.}$$

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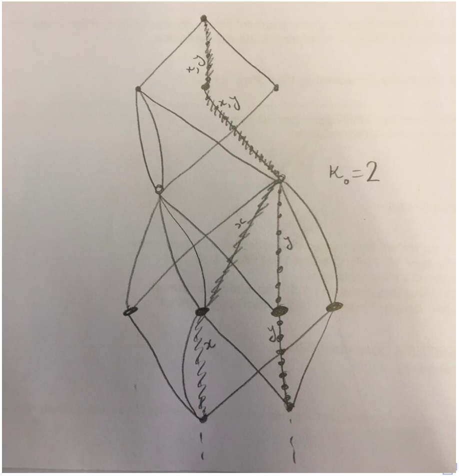
So each finite path of depth k is a neighbourhood around the (points) all infinite paths that their initial k edges are that finite path.

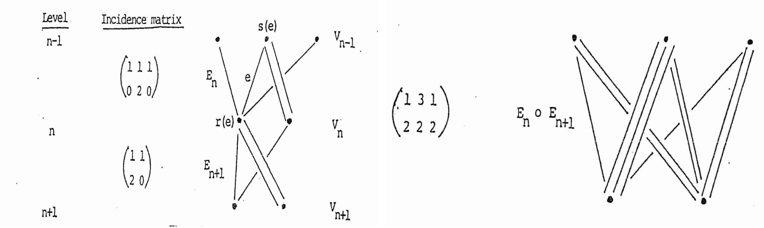

x,y
x,y
$K_0=2$
x
y
y
x

Telescoped Bratteli diagram.

Consider (B, V) and fix a sequence $(n_i)_{i\geq 1}$ and consider a new diagram $B' = (V', E')$ where

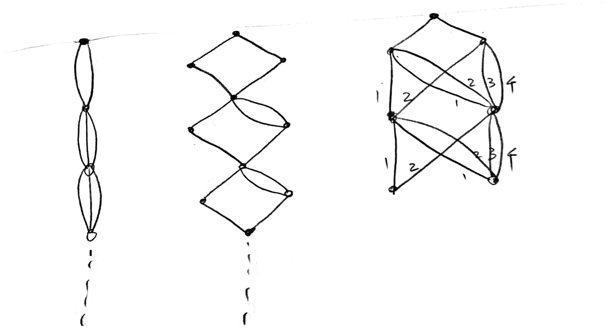
$$M'_i = M_{n_i} \times M_{n_i - 1} \times \cdots \times M_{n_{i-1}+1}$$

This new diagram is called a telescoped diagram from $B = (V, E)$.



Equivalent Bratteli diagrams.

- Two Bratteli diagrams $B = (V, E)$ and $B' = (V', E')$ are called equivalent if they can be constructed by telescoping of a third diagram along two subsequences.



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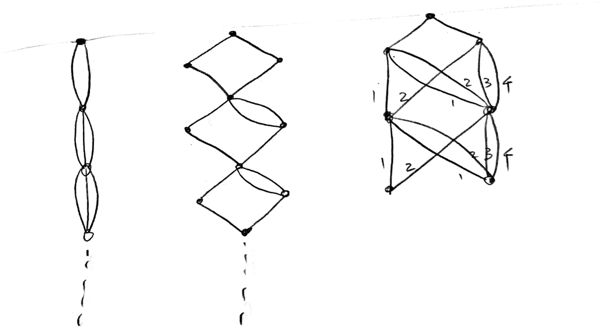
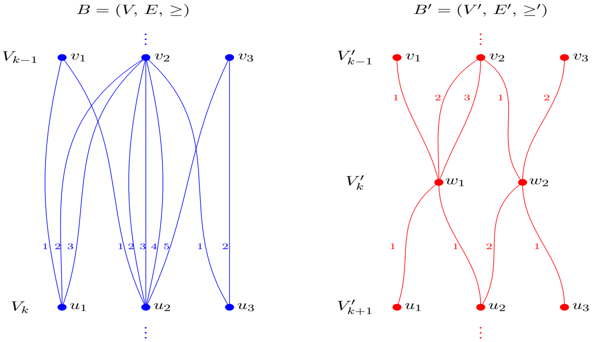


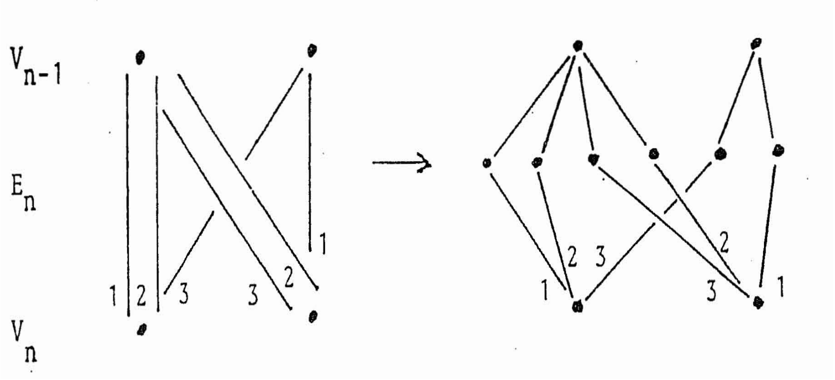
- In particular, Every telescoped form of a Bratteli diagram $B = (V, E)$ is equivalent to it.

Microscoping.



Symbol Splitting.

This is a kind of microscoping that gives us an equivalent Bratteli diagram.



Stationary Bratteli Diagram.

The Bratteli diagram $(B, V, \leq)$:

- is called of finite rank d If there exists a telescoped form of it such that

$$\forall\ i \geq 1 : \quad |V_i| \text{ is the constant } d.$$

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$$M_i = M.$$

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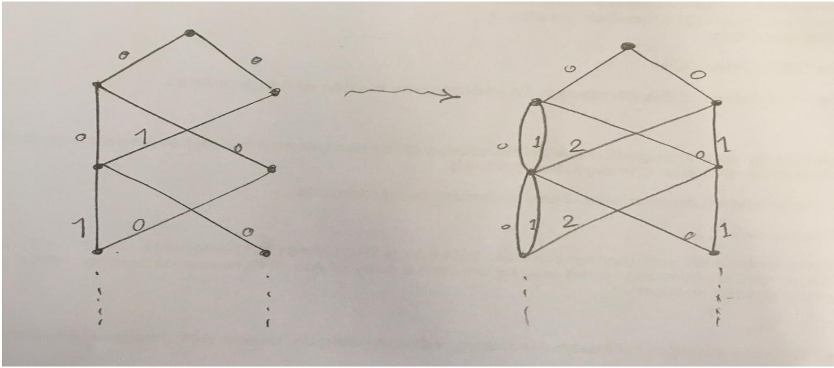
$$M_i = M.$$
- is called stationary ordered if it is stationary and when we fix some $j \in \{1, \cdots, d\}$ then
$$\forall\ i \geq 1: \text{ the partial orderings on } r^{-1}(v_j^{(i)}) \text{ are the same.}$$

Example.



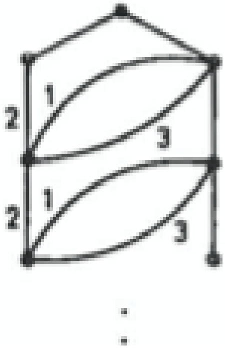
Simple Bratteli diagrams..

- The Bratteli diagram $B = (V, E)$ is called simple if there exists a telescoped of it, say $B' = (V', E', \{M_i'\}_i)$, such that

$$\forall i \geq 1 : \; M_i' > 0.$$

It means that for every two different vertices u, v in two different levels k, ℓ, there exists at least one finite path between these two levels that connects u to v.

Regardless of the ordering, the following diagram is not simple.



This implies that X_B, equipped with the above topology, does not have any isolated point and therefore, it is a Cantor set.

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- An ordered diagram $B = (V, E, \leq)$ is called simple ordered if $B = (V, E)$ is simple and with its ordering it has a unique infinite min path and a unique infinite max path. i.e. there are exactly two infinite paths $(e_1, e_2, \ldots) \neq (f_1, f_2, \ldots)$ such that

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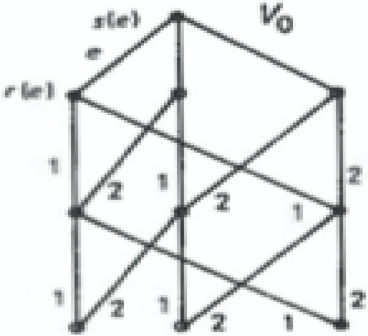
- *Remark. In some literatures, an ordered Bratteli diagram which has unique minimal and maximal infinite paths is called properly ordered. Note that such a diagram may not necessarily be simple.*

With respect to the ordering, the following diagram is not simple.



Vershik system.

Consider a simple ordered Bratteli diagram $B = (V, E, \leq)$ and the Cantor set X_B. Define a map $T_B : X_B \to X_B$ with

- $T_B(x_{\max}) = x_{\min}$.

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- For every point $x_{\max} \neq x = (e_1, e_2, \dots) \in X_B$ if i_0 is the first i that $e_i \notin E_{\max}$ then this edge has a successor (between all the edges with the range $r(e_{i_0})$). Let's call its successor by f_{i_0} then

$$T_B(e_1, \dots, e_{i_0}, \dots) = (e_{\min}, e_{\min}, \dots, e_{\min}, f_{i_0}, e_{i_0+1}, \dots).$$

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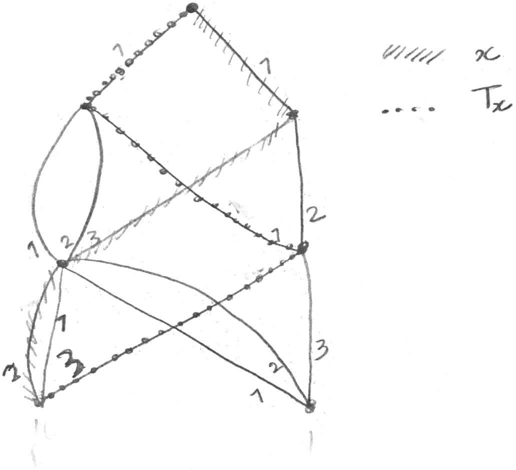
$$T_B(e_1, \dots, e_{i_0}, \dots) = (e_{\min}, e_{\min}, \dots, e_{\min}, f_{i_0}, e_{i_0+1}, \dots).$$

(X_B, T_B) is called the Vershik system on $(B, \leq)$.


1
1
1
2
3
2
1
1
3
3
2
1
x
Tx

It is easy to see that

- T_B is a homeomorphism. (Exercise.)
- (X_B, T_B) is a minimal system. i.e. every point (infinite path) has a dense orbit. (Exercise.)

You wanna know some examples?

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Proposition

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To prove this we need to know Kakutani-Rokhlin partitions of minimal Cantor systems. For doing that we have to recall the notion of first return time map.

First Return Time map.

Let (X, T) be a minimal Cantor system and $U \overset{\text{clopen}}{\subseteq} X$. For every $x \in X$ the the first return time map is defined by

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In some literatures this system (U, T_U) is called the derivative of (X, T) on U.