Bratteli-Vershik Models of Cantor Minimal Systems

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K-R partitions of Odometers.

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$$\mathcal{B}_2 = \{[00], [10], [01], [11]\}.$$

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$$[000] \mapsto [100] \mapsto [010] \mapsto [110] \mapsto [001] \mapsto [101] \mapsto [011] \mapsto [111] \mapsto [000].$$

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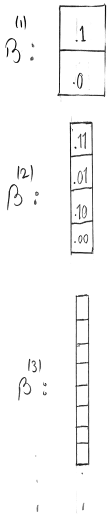
β(1):
.1
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β(2):
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.01
.10
.00
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So $(Z, \imath)$ is in fact an inverse limit system:

$$\mathbb{Z}_{(2^n)} = \{(x_n)_n \in \prod_{n \in \mathbb{N}} \mathbb{Z}/2^n\mathbb{Z} : \ \phi_n(x_n + 1) = x_n\}$$

General Case.

- Similar arguments work for the general case:

$$\mathbb{Z}_{(p_n)} = \{(x_n)_n \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} : \ \phi_n(x_n+1) = x_n\}$$

where $p_n | p_{n+1}$ and ϕ_n are the canonical homomorphisms from $\mathbb{Z}/p_{n+1}\mathbb{Z} \to \mathbb{Z}/p_n\mathbb{Z}$. In fact,

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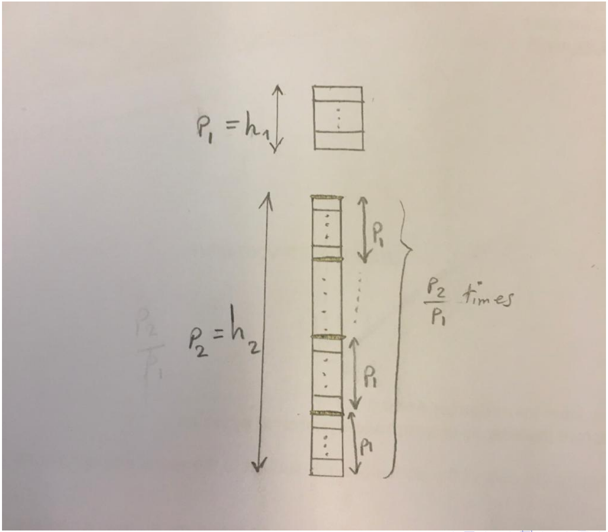
In this case the heights of the unique tower in $\mathcal{B}^{(n)}$ is equal to p_n.

The associated K-R partitions.



Vershik systems conjugate to a given Cantor minimal system.

For a given C.M.S. (X, T) and any point $x_0 \in X$, by having a refining sequence of K-R partitions that their bases converge to $\{x_0\}$ one can define a simple ordered Bratteli diagram $B = (V, E, \leq)$ so that the Vershik system (X_B, T_B) is conjugate to (X, T).

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- The number of vertexes at level n, $n \geq 1$ of $B = (V, E, \leq)$ is the same as the number of towers in the n'th K-R partition of (X, T). For the first level $V_0 = \{v_0\}$ represents the trivial tower $\mathcal{B}_0 = X$.

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- the number of edges between $v_k^{(n)} \in V_n$ and $v_\ell^{(n-1)} \in V_{n-1}$ is the same as the number of times that the tower $T_\ell \in \mathcal{B}_{n-1}$ occurs as a subtower of $T_k \in \mathcal{B}_n$. In other words,

$$\exists e \in E_n;\ r(e) = v_k^{(n)} \text{ and } s(e) = v_\ell^{(n-1)} \text{ iff } \exists\, 0 \leq j \leq h_k;\ T^j(B_k^{(n)}) \subset B_\ell^{(n-1)}$$

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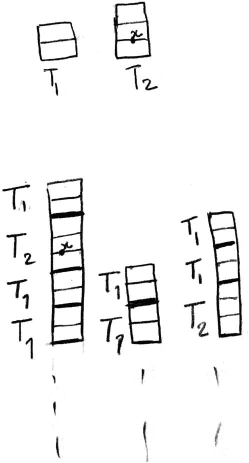
- For each $v_k^{(n)} \in V_n$ that is the representative of the $T_k \in V_n$, the ordering on E_v is considered to be exactly the same as the ordering of occurrences of the towers of $\mathcal{B}_{n-1}$ as sub-towers of T_k. Note that by minimality every tower of $\mathcal{B}_{n-1}$ will appear as a sub-tower of T_k.
- Each infinite path on $B = (V, E, \leq)$ is associated with

$$\bigcap_{n \geq 1} T^{j_n}(B_k^{(n)})$$

so that for each n the index k belongs to $[1, |V_n|] = [1, = |\mathcal{B}_n|]$.

- For each $v_k^{(n)} \in V_n$, the min edge is the representative of the base of the tower $T_k \in \mathcal{B}_n$ and since the intersection of the bases of the towers converge to a unique point x_0, there exists a unique infinite min path $x_{\min}$ in $B = (V, E, \leq)$ that associates with

$$\bigcap_{n \geq 1} (B_k^{(n)}).$$


T1
T2
x
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T1
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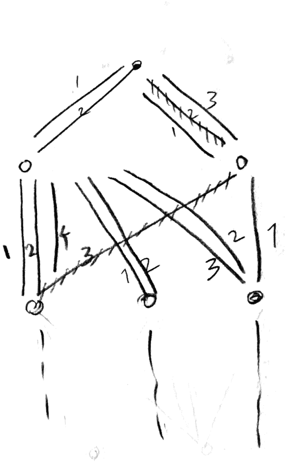
1
2
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1
1
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1
2
3
2
1

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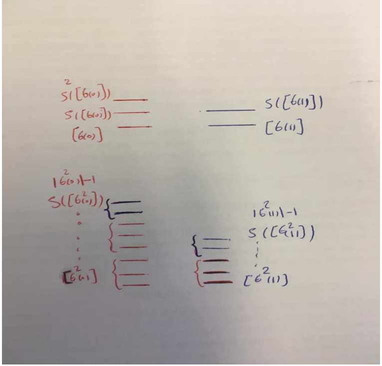
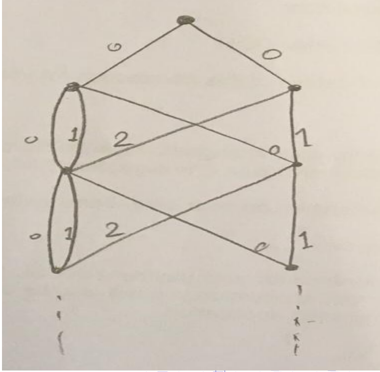
- So by the above construction, the Vershik map on the diagram is just moving between consecutive edges of the diagrams that are representatives of consecutive levels of towers in K-R partitions. In fact, if $x = (e_1, e_2, \dots)$ is an infinite path and e_{i_0} is the first edge that does not belong to $E_{\max}$, if we telescope the diagram from zero level to level i_0 to have $B' = (V', E', \leq)$, the new set of edges between v_0 and $v_s \in V'_1 = V_{i_0}$ are indeed the consecutive levels of the tower $T_s \in \mathcal{B}_{i_0} = \mathcal{B}'_1$.

Examples.

- Fibonacci Sturmian system. generated by the substitution: $\sigma : \{0,1\} \to \{0,1\}^*$ that

$$\sigma(0) = 001, \quad \sigma(1) = 01.$$





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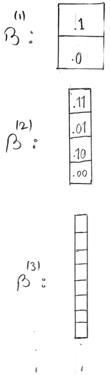

Vershik system as an inverse limit system.

Consider the Vershik system (X_B, T_B) on the simply ordered Bratteli diagram $B = (V, E, \leq)$. For each $k \geq 1$, Let

$$\Sigma_k = \{(e_1, e_2, \ldots, e_k) : \; e_i \in E_i\}$$

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$$\tilde{\pi}_k(x) := (\pi_k(T_B^n(x))_{n\in\mathbb{Z}}.$$

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Consider the topology induced from X_B on $\tilde{\pi}_k(X_B)$.

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one can say that (Y_k, S) is a subshift system (maybe a trivial one) with alphabet Σ_k.

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Consequently, we have a sequence of topological factors $\{(Y_k, S)\}_{k\geq 1}$ that they "converge" to (X_B, T_B). In fact

(X_B, T_B) is the inverse limit of subshift systems $\{(Y_k, S)\}_{k\geq 1}$.

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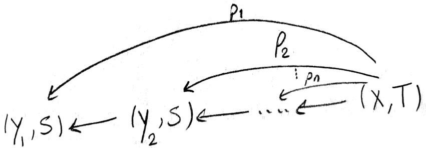
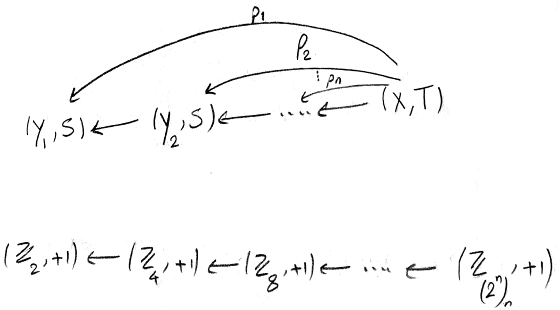


$$(\mathbb{Z}_2, +1) \leftarrow (\mathbb{Z}_4, +1) \leftarrow (\mathbb{Z}_8, +1) \leftarrow \cdots \leftarrow (\mathbb{Z}_{(2^n)_n}, +1)$$



Exercise. If (X, T) is expansive then

$$\exists\ k_0 \text{ such that } \forall k > k_0 : \quad (X, T) \text{ is conjugate to } (Y_k, S).$$

Application to Strong Orbit Equivalence.

Every two Vershik systems on equivalent Bratteli diagrams are Strongly orbit equivalent.

Orbit Equivalence.

- Two dynamical systems (X, T) and (Y, S) are said to be orbit equivalent if there exists a homeomorphism $h : X \to Y$ such that

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Obviously this is an equivalence relation between dynamical systems.

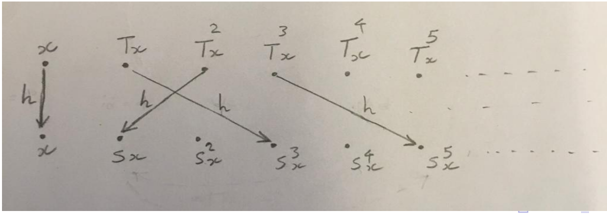
Orbit Equivalence.

- Two dynamical systems (X, T) and (Y, S) are said to be orbit equivalent if there exists a homeomorphism $h : X \to Y$ such that

$$\forall x \in X : \quad h(\mathcal{O}_T(x)) = \mathcal{O}_S(h(x)).$$

Obviously this is an equivalence relation between dynamical systems.

Remark. By the above definiton, the set of points of each orbit is preserved by h.



Orbit Equivalence.

In other words,

$$\exists\ \alpha : X \to \mathbb{Z}, \quad \beta : Y \to \mathbb{Z}$$

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These are called the the cocycle maps associated with T and S respectively. Considering the isomorphism between (Y, S) and $(X, h^{-1} \circ S \circ h)$ we can assume that $X = Y$ and so

$$T(x) = S^{\alpha(x)}(x), \quad S(x) = T^{\beta(x)}(x).$$

- So $T^k(x) = S^{\ell(k,x)}(x)$ where $\ell : \mathbb{Z} \times X \to \mathbb{Z}$:

$$\begin{aligned} \ell(k,x) &= \alpha(x) + \alpha(Tx) + \cdots + \alpha(T^{k-1}x) : \quad k > 0 \\ \ell(k,x) &= -[\alpha(T^{-1}x) + \alpha(T^{-2}x) + \cdots + \alpha(T^{k}x)] : \quad k < 0 \\ \ell(0,x) &= 0. \end{aligned}$$

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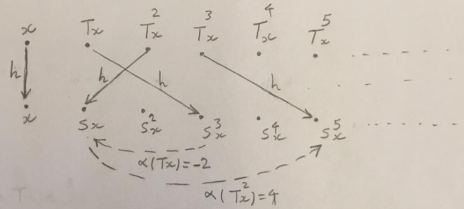
- Clearly, minimality is preserved by orbit equivalence.
- when (X, T) is minimal then the two cocycle maps α and β are uniquely determined on X and Y. (exercise)



$$T^3x = Sx \Rightarrow \alpha(x) = 3 \qquad T^2x = Sx \Rightarrow \ell(2, x) = 1$$

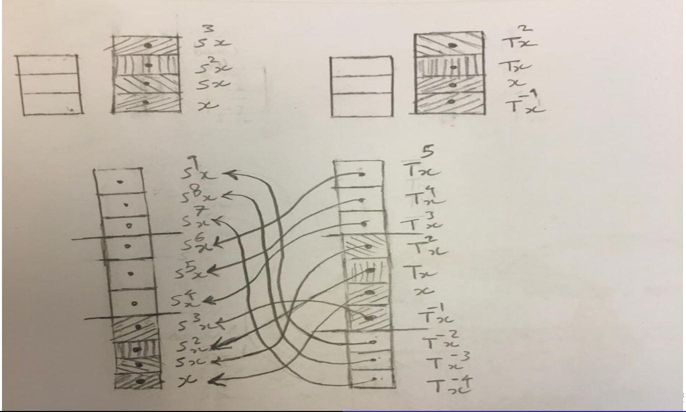
$$1 = \ell(2, x) = \alpha(x) + \alpha(Tx) = 3 + \alpha(Tx) \Rightarrow \alpha(Tx) = -2$$

(*) $\alpha(Tx) = -2$ shows that: From the image of Tx under the map h, we should jump backward twice under S to reach the image of T^2x under the map h.

(*) similar explanation is applied to justify $\alpha(T^2x) = 4$.

Examples.

Every two Vershik systems on a simple Bratteli diagram (B, V) are orbit equivalent.



Full characterization of orbit equivalence.

Theorem

For the wo Cantor minimal systems (X, T) and (Y, S) TFAE

- (X, T) *is orbit equivalent to* (Y, S).
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Proof. $\Rightarrow$ Suppose that there exists a homeomorphism $F : X \to X$ that preserves the orbits. Then

$$T(x) = S^{\alpha(x)}(x), \quad S(x) = T^{\beta(x)}(x).$$

Note that α and β are Borel functions (exercise). In fact,

$$\forall k \in \mathbb{Z} : \quad A_k = \{x \in A : \ \alpha(x) = k\}$$

are closed sets and the family of all these closed sets make partitions for every Borel set E.

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which means that μ is T-invariant as well. Similrly every T-invariant measure ν will be an invariant measure for S.

For the proof of $\Rightarrow$ we need more algebraic arguments and tools that we have not seen them in this course. Look at the paper of T.Giordano, I. Putnam and C. Skau, 1995.

Cocycles and continuity

As the two cocycle maps are integer valued, when the two orbit equivalent systems are on the Cantor set, we can discuss about continuity of these cocyles.

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- If (X, T) and (X, S) are two Cantor minimal systems with the same sets of orbits, if the cocycle maps are continuous (and therefore uniformly bounded) then the two systems are flip-conjugate.

  flip conjugacy: S is conjugate to T or to T^{-1}.

Strong orbit equivalence.

It may happen that the cocycle maps are continuous every where except exactly finitely many points. To know about the case that the set of discontinuity points are finite and more than one, we will refer you to the paper by GPS-1995.

- Tow Cantor minimal systems are called strongly orbit equivalent if the two cocycle maps are discontinuous at exactly one point in X.

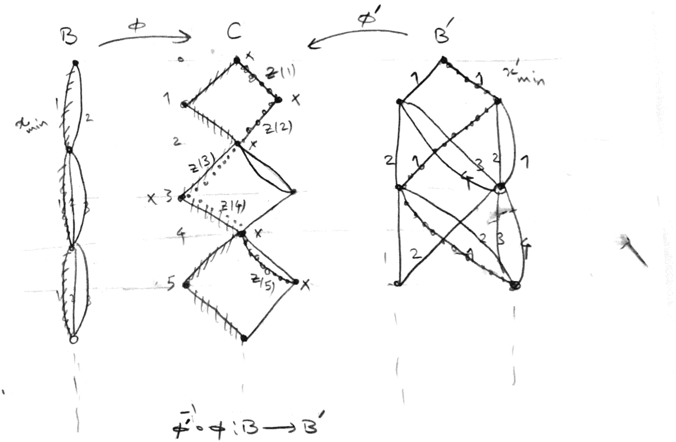
so in this case each of the two cocycles are tending to infinity around exactly one point in X.

strong orbit equivalence Vershik systems.

Theorem (GPS, 1995)

Two Vershik systems on equivalent Bratteli diagrams $B = (V, E, \leq)$ *and* $B' = (V', E', \leq)$ *are* strongly orbit equivalent*.*

In particular, Two Vershik systems associated with two different orderings on the same Bratteli diagrams $B = (V, E)$ are strongly orbit equivalent.


B
φ
C
φ'
B'
x_{min}
x'_{min}
z(1)
z(2)
z(3)
z(4)
z(5)
$\phi'^{-1} \circ \phi : B \to B'$