Bratteli-Vershik Models of Cantor Minimal Systems

Maryam Hosseini

CIMPA School, Bejaia, Algeria

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there exists a Bratteli-Vershik system (X_B, T_B) that is conjugate to that.

To show this we need Kakutani-Rokhlin partitions that are based on the notion of first return time to a clopen set.

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$n(x)$ exists for every x as the system is minimal. In fact, the above sets have bounded gaps. This was because:

$$\forall \text{ clopen } U \subset X, \ \exists \ N; \ \cup_{n=0}^{N} T^{-n}(U) = X.$$

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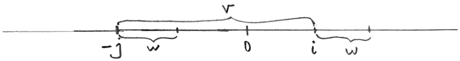

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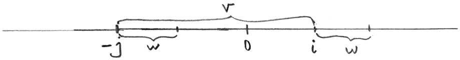

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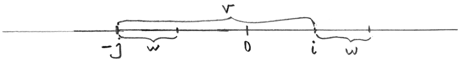

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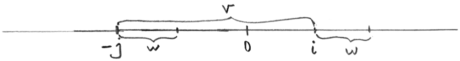

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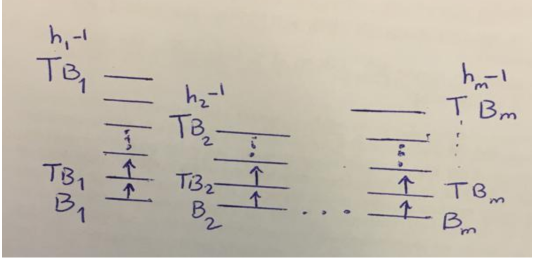
So for $U = [w]$ and x:

$$n(x) = \min_{1 \le i \le m} |v_i|.$$

Kakutani-Rokhlin partitions.

Let (X, T) be a minimal Cantor system where T is a homeomorphism. A Kakutani-Rokhlin partition, say $\mathcal{P}$, of (X, T) is a collection of towers of the form:



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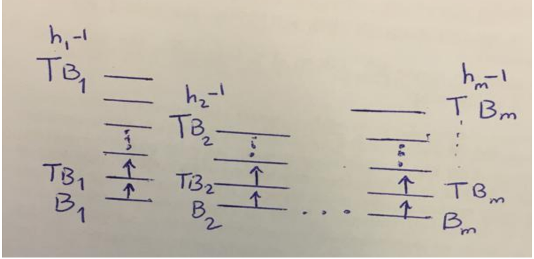

In fact,

$$\mathcal{P} = \{T^j(B_i) : \ 1 \leq i \leq m, \ 1 \leq j \leq h_i\}$$

where $\{B_i\}_{i=1}^m$ are disjoint clopen subsets of X and $h_i \in \mathbb{N}$.

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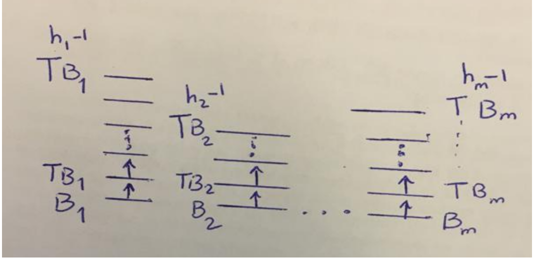

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where $\{B_i\}_{i=1}^m$ are disjoint clopen subsets of X and $h_i \in \mathbb{N}$.

$$(T^{h_i} B_i) \subseteq \cup_{i=1}^m B_i, \qquad X = \cup_{i=1}^m \cup_{j=1}^{h_i} T^j(B_i).$$

- $B = \cup_{i=1}^{m} B_i$ is called the basis of partition.
- for each $1 \leq i \leq m$, $\{B_i, TB_i, \dots, T^{h_i} B_i\}$ is called a tower. So the partition is a finite union of towers.
- For each $1 \leq i \leq m$ the first return time to B for the points of B_i is h_i
- The map T is defined explicitly every where except the top levels of the towers. We just know that the top level will be mapped to B but where exactly? we don't know.

Example, using return words.

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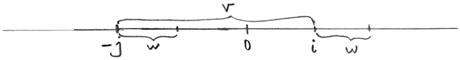


In fact, by minimality the set of return words to w is finite:

$$\mathcal{R}(w) = \{v_1, v_2, \ldots, v_m\} : \quad m < \infty.$$

So for each $v_i \in \mathcal{R}(w)$, $i = 1, \ldots, m$ we have a tower above that.

...

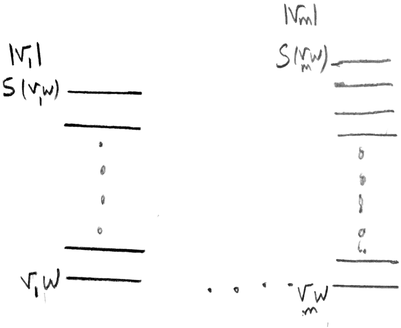
So let $B = \cup_{i=1}^{m} [V_i]$,

$$\mathcal{T}_i = \{S^j([v_i]) : \ v_i \in \mathcal{R}(w), \ 0 \leq j \leq |v_i|\}, i = 1, \ldots, m$$

are the towers and

$$\mathcal{B} = \{\mathcal{T}_i\}_{i=1}^{m}$$

is a Kakutani-Rokhlin partition for (X, S).



Example.

Consider the two sided subshift $(\overline{\mathcal{O}(x)}, S)$ generated by the primitive substitution:

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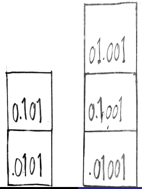
Let $U = [w]$, with $w = 01$, as a clopen set around the fixed point of the substitution: Then

$$\mathcal{R}(w) = \{01, 010\}$$

Therefore, $\mathcal{B} = \{[0101], S[(0101]), [01001], S([01001]), S^2([01001])\}$ is a K-R partition of (X, S).



Nested K-R partitions.

The K-R Partition $\mathcal{B}'$ with basis B' is called nested in the K-R partition $\mathcal{B}$ with basis B if $B' \subset B$ and $\mathcal{B}'$ is a refinement of $\mathcal{B}$, i.e.

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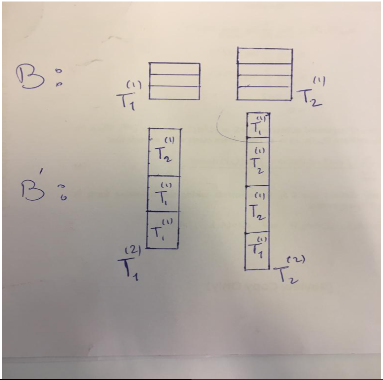
Roughly speaking, we cut and stack the towers of $\mathcal{B}$ to create $\mathcal{B}'$.



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- $\bigcap_n B^{(n)} = \{x\}$.
- The sequence $\{\mathcal{B}^{(n)}\}_{n\geq 0}$ is nested, i.e. $\mathcal{B}^{(n+1)}$ is nested in $\mathcal{B}^{(n)}$ for every n.
- $\bigcup_n \mathcal{B}^{(n)}$ generates the topology of X.

K-R partitions of minimal Cantor systems.

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$$\forall x \in C : \ n_C : X \to \mathbb{Z}; \ n_C(x) = \inf\{n > 0 : \ T^n x \in C\}.$$

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$$\forall x \in C: \; n_C : X \to \mathbb{Z}; \; n_C(x) = \inf\{n > 0 : \; T^n x \in C\}.$$

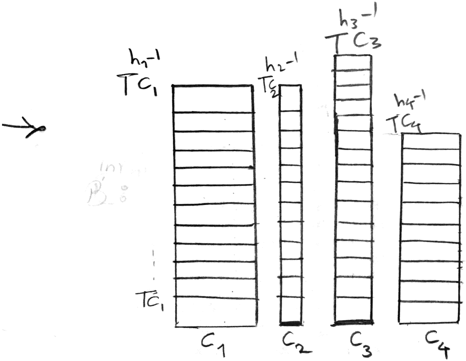
Since (X, T) is minimal, n_C is well-defined and continuous and in fact,

$$C = \cup_{i=1}^{m} C_i; \;\; \forall x \in C_i : \;\; n_C(x) = h_i \in \mathbb{N}.$$

...

Let's define a partition by the above subsets C_i, $i = 1, \ldots, m$.

$$\mathcal{B}^{(1)} := \{T^j C_i : \quad 0 \le j \le h_i - 1, 1 \le i \le m\}.$$



- every two cells in $\mathcal{B}^{(1)}$ are disjoint. This is just because of definition of n_C and T being a homeomorphism. (exercise)

- $\mathcal{B}^{(1)}$ covers X. Because

$$W = \bigcup_{i=1}^{m} \bigcup_{j=1}^{h_i - 1} C_{ij} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{h_i - 1} T^j C_i$$

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Consequently, $\mathcal{B}^{(1)}$ is a K-R partition for (X, T). Now we show that we can have $\mathcal{B}^{(1)}$ finer than the partition $\mathcal{P}_1$.

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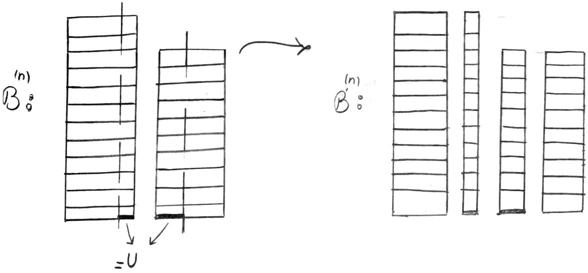

Suppos that there exists some $U \in \mathcal{P}_1$ so that it has intersection with C_ij but C_ij is not contained in U. As the number of elements in $\mathcal{B}^{(1)}$ is finite, consider the least diameter δ of intersections of C_ij's with the elements of $\mathcal{P}_1$ and cut the towers of $\mathcal{B}^{(1)}$ into finitely many thiner towers (with diameter δ) that each cell of these new towers are contained in an element of $\mathcal{P}_1$.

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By similar arguments as for $\mathcal{B}^{(1)}$, one can deduce that $\mathcal{B}^{(2)}$ is a K-R partition. Now we show that this is finer than $\mathcal{B}^{(1)}$ and could be constructed to be finer than $\mathcal{P}_2$ as well.

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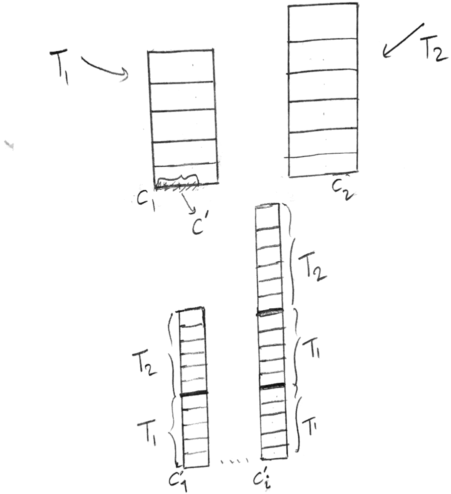
- About $\mathcal{B}^{(1)}$: Since $C' \subset C_1$, the initial h_1 levels of all the towers in $\mathcal{B}^{(1)}$ are arranged in the same shape as the first tower of $\mathcal{B}^{(1)}$ with base C_1. And since $T^{h_1}(C') \subset C_i$ for some other $i = 1, \ldots, m$ the next h_i levels of each tower of $\mathcal{B}^{(2)}$ is arranged in the same way as the associated i'th tower in $\mathcal{B}^{(1)}$ and in fact, each of these levels is contained in a unique level of the associated i'th tower in $\mathcal{B}^{(1)}$.



T_1
T_2
C_1
C'
C_2
T_2
T_1
T_1
T_2
T_1
C'_1
C'_i

...

By continuing the above procedure inductively by choosing nested sequence of clopen subsets of C_1 around x_0 and constructing towers based on them and so a sequence of refining K-R partitions that each of them refines a new $\mathcal{P}_n$ (increasingly by n), we will have the intersection of all the bases of the towers to be $\{x_0\}$.

Example.

Let's get back to the primitive substitution:

$$\sigma : \; 0 \mapsto 001, \quad 1 \mapsto 01$$

and $C = [w]$, $w = 0$, as a clopen set around the fixed point of the substitution:

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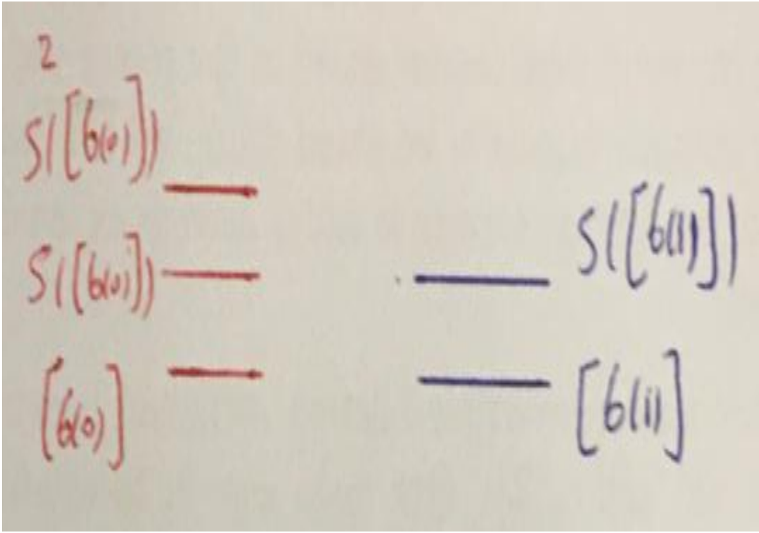
Then

$$\mathcal{R}(w) = \{0, 01\}, \quad [0] = [00] \cup [01] = [001] \cup [01] = [\sigma(0)] \cup [\sigma(1)].$$

So we should have two towers with disjoint bases [001] and [01]. So

$$\begin{aligned} \mathcal{B}^{(1)} &= \{[\sigma(0)], S([\sigma(0)]), S^2([\sigma(0)]), [\sigma(1)], S([\sigma(1)])\} \\ &= \{T_1^{(1)}, T_2^{(1)}\} \end{aligned}$$


2
S([b(0)])
S([b(0)])
[b(0)]
S([b(1)])
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Now let $C'_1 = [w'], w' = 001$. Then

$\mathcal{R}(w) = \{001, 00101\}$, $[001] = [00100101] \cup [00101] = [\sigma^2(0)] \cup [\sigma^2(1)]$

So we should have two towers with disjoint bases [001001] and [00101]:

$$\begin{aligned}\mathcal{B}^{(2)} &= \{[\sigma^2(0)], \ldots, S^7([\sigma^2(0)]), [\sigma^2(1)], \ldots, S^4([\sigma^2(1)])\} \\ &= \{T_1^{(1)}, T_1^{(1)}, T_2^{(1)}, T_1^{(1)}, T_2^{(1)}\} \\ &= \{T_1^{(2)}, T_2^{(2)}\}\end{aligned}$$

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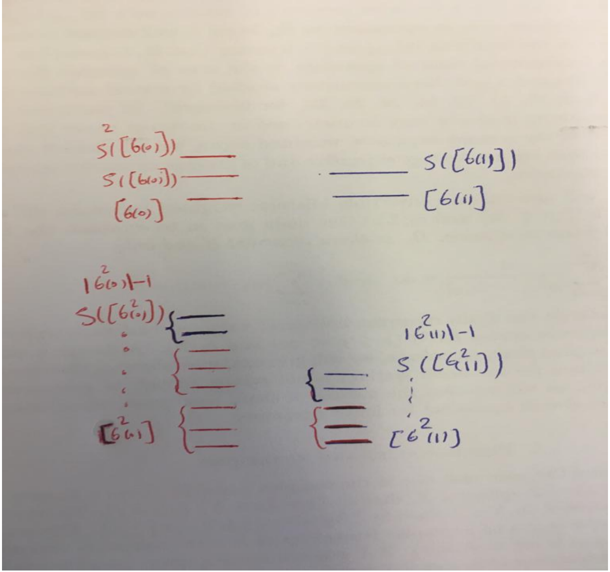
\end{aligned}$$

If you continue this procedure inductively, you will get

$$\begin{aligned}
\mathcal{B}^{(n)} &= \{S^j\sigma^n([a]) : \ a = 0, 1, \ \ 0 \le j < |\sigma^n(a)|\} \\
&= \{T_1^{(n-1)}, T_1^{(n-1)}, T_2^{(n-1)}, T_1^{(n-1)}, T_2^{(n-1)}\} \\
&= \{T_1^{(n)}, T_2^{(n)}\}
\end{aligned}$$



$S^2([G(0)])$
$S([G(0)])$
$[G(0)]$
$S([G(1)])$
$[G(1)]$
$|G^2(0)|-1$
$S([G^2(0)])$
$[G^2(0)]$
$|G^2(1)|-1$
$S([G^2(1)])$
$[G^2(1)]$

General case.

Proposition

For every aperiodic two sided minimal subshift (X, S) associated to a primitive proper substitution $\sigma : A \to A^*$ that A is a finite alphabet,

$$\mathcal{B}^{(n)} = \{S^j \sigma^n([a]) : \; a \in A, 0 \leq j < |\sigma^n(a)|\}, \; n \geq 0$$

is a refining sequence of K-R partitions.

Remark. Proper means that there are letters $r, \ell \in A$ that for every $a \in A$, $\sigma(a)$ starts with r and ends up with ℓ.

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To prove this proposition knowing the notion of recognizability will ease it.

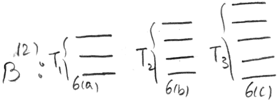
For instance:

$\sigma : \{a, b, c\} \longrightarrow \{a, b, c\}^*$

$\sigma(a) = abc$

$\sigma(b) = abbc$

$\sigma(c) = abccc$



$$\begin{cases} \sigma^2(a) = (abc)(abbc)(abccc) \\ \sigma^2(b) = (abc)(abbc)(abbc)(abccc) \\ \sigma^2(c) = (abc)(abbc)(abccc)(abccc)(abccc) \end{cases}$$

$$\begin{cases} \sigma^2(a) = T_1 T_2 T_3 \\ \sigma^2(b) = T_1 T_2 T_2 T_3 \\ \sigma^2(c) = T_1 T_2 T_3 T_3 T_3 \end{cases}$$