Growth-fragmentation equations

Exercice 1[Non-existence for the eigenvalue problem] Consider the growth fragmentation equation with the growth rate $\tau(x) = 1$, the division rate

$$B(x) = \begin{cases} 0, & 0 \le x < 1, \\ b, & 1 \le x < 2, \\ 0, & 2 \le x, \end{cases}$$

for some $b > 0$, and the mitosis kernel $\kappa(z) = \delta(z - 1/2)$. The goal is to prove that for $b < \ln(2)$ there is no solution to the eigenproblem.

a) Write the eigenproblem, i.e. the equation whose solution is (λ, N).

b) Prove that if there exsits a solution $(\lambda, N) \in \mathbb{R}^+ \times L^1(\mathbb{R}^+)$ to the eigenproblem, then, N is continuous.

c) Find an expression of N for $x \ge 2$ and for $x \le \dfrac{1}{2}$.

d) Find an expression of N for $1 \le x \le 2$.

e) Deduce that for $\dfrac{1}{2} \le x \le 1$, we have

$$N(x) = 2be^{-\lambda x} \int_1^{2x} e^{\frac{\lambda u}{2}} N(u)du.$$

f) Prove that the eigenproblem has a solution provided that there exsits $\lambda > 0$ such that $G(\lambda) = 0$, where

$$G(\lambda) = \lambda + 2b\left(e^{-\lambda/2} - e^{-\lambda - b}\right).$$

g) Calculate $G(0)$ and $G'(\lambda)$ and conclude.

Exercice 2[Monotony of the Malthus parameter] Consider the growth fragmentation equation with the two sets of parameters $(\tau = \tau_1, B = \tau_1\beta, k)$ and $(\tau = \tau_2, B = \tau_2\beta, k)$. We assume that

$$\tau_1(x) \le \tau_2(x), \qquad x \in \mathbb{R}^+$$

The goal is to prove that the associated Malthus parameters λ_1 and λ_2 verify

$$\lambda_1 \le \lambda_2.$$

We introduce the notation

$$\langle f, g\rangle = \int_{\mathbb{R}^+} f(x)g(x)dx, \qquad f \in L^1(\mathbb{R}^+),\ g \in L^\infty(\mathbb{R}^+).$$

Calculate $\lambda_1\langle N_1, \Phi_2\rangle$ and conclude.

Exercice 3[Asymptotic behaviour for the pure fragmentation equation] Consider the *pure* fragmentation equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = -B(x)u(t,x) + 2\displaystyle\int_x^\infty B(y)\kappa\left(\frac{x}{y}\right) u(t,y)\frac{dy}{y}, \\ u(0,x) = u_0(x), \qquad u_0 \in L^1(\mathbb{R}^+). \end{cases} \tag{1}$$

where $B(x) = \alpha x^\gamma$ for $\alpha > 0, \gamma > 0$, $\kappa \in \mathcal{M}^+(\mathbb{R}^+)$ such as

$$\int_0^1 \kappa(z)dz = 1, \qquad \int_0^1 z\kappa(z)dz = \frac{1}{2}.$$

a) Show that fragmentation equation can be turned into a growth-fragmentation equation with the change of variables $(t = t, y = t^{1/\gamma} x)$.

b) Deduce the asymptotic behaviour of the solution u to (**??**).