Exercice 1 | Non-existence for the eigenvalue problem] Consider the growth fragmentation equation with the growth rate $\tau(x) = 1$, the division rate

$$B(x) = \begin{cases} 0, & 0 \leq x < 1, \\ b, & 1 \leq x < 2, \\ 0, & 2 \leq x, \end{cases}$$

for some $b > 0$, and the mitosis kernel $\kappa(z) = \delta(z - 1/2)$. The goal is to prove that for $b < \ln(2)$ there is no solution to the eigenproblem.

a) Write the eigenproblem, i.e. the equation whose solution is $(\lambda, N)$.

b) Prove that if there exists a solution $(\lambda, N) \in \mathbb{R}^+ \times L^1(\mathbb{R}^+)$ to the eigenproblem, then, $N$ is continuous.

c) Find an expression of $N$ for $x \geq 2$ and for $x \leq \frac{1}{2}$.

d) Find an expression of $N$ for $1 \leq x \leq 2$.

e) Deduce that for $\frac{1}{2} \leq x \leq 1$, we have

$$N(x) = 2be^{-\lambda x} \int_1^{2x} e^{\lambda u} N(u) du.$$

f) Prove that the eigenproblem has a solution provided that there exists $\lambda > 0$ such that $G(\lambda) = 0$, where

$$G(\lambda) = \lambda + 2b \left( e^{-\lambda/2} - e^{-\lambda/b} \right).$$

g) Calculate $G(0)$ and $G'(\lambda)$ and conclude.

Exercice 2 | Monotony of the Malthus parameter] Consider the growth fragmentation equation with the two sets of parameters $(\tau = \tau_1, B = \tau_1 \beta, k)$ and $(\tau = \tau_2, B = \tau_2 \beta, k)$. We assume that

$$\tau_1(x) \leq \tau_2(x), \quad x \in \mathbb{R}^+$$

The goal is to prove that the associated Malthus parameters $\lambda_1$ and $\lambda_2$ verify

$$\lambda_1 \leq \lambda_2.$$

We introduce the notation

$$\langle f, g \rangle = \int_{\mathbb{R}^+} f(x)g(x) dx, \quad f \in L^1(\mathbb{R}^+), \; g \in L^\infty(\mathbb{R}^+).$$

Calculate $\lambda_1 \langle N_1, \Phi_2 \rangle$ and conclude.
Exercise 3: Asymptotic behaviour for the pure fragmentation equation. Consider the pure fragmentation equation

\[
\begin{aligned}
    \frac{\partial}{\partial t} u(t,x) &= -B(x)u(t,x) + 2 \int_x^\infty B(y)\kappa\left(\frac{x}{y}\right) u(t,y) \frac{dy}{y}, \\
    u(0,x) &= u_0(x), \quad u_0 \in L^1(\mathbb{R}^+).
\end{aligned}
\]

where \( B(x) = \alpha x^\gamma \) for \( \alpha > 0, \gamma > 0 \), \( \kappa \in \mathcal{M}^+(\mathbb{R}^+) \) such as

\[
\int_0^1 \kappa(z)dz = 1, \quad \int_0^1 z\kappa(z)dz = \frac{1}{2}.
\]

a) Show that fragmentation equation can be turned into a growth-fragmentation equation with the change of variables \( (t = t, y = t^{1/\gamma} x) \).

b) Deduce the asymptotic behaviour of the solution \( u \) to (??).